The Fourier Transform and its Applications

The Fourier Transform:

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi sx} dx$$

The Inverse Fourier Transform:

$$f(x) = \int_{-\infty}^{\infty} F(s) e^{i2\pi sx} ds$$

Symmetry Properties:
If $g(x)$ is real valued, then $G(s)$ is Hermitian:

$$G(-s) = G^*(s)$$

If $g(x)$ is imaginary valued, then $G(s)$ is Anti-Hermitian:

$$G(-s) = -G^*(s)$$

In general:

$$g(x) \ = \ e(x) + o(x) \ = \ e_R(x) + ie_I(x) + o_R(x) + io_I(x)$$
$$G(s) = E(s) + O(s) = E_R(s) + iE_I(s) + iO_I(s) + O_R(s)$$

Convolution:

$$(g * h)(x) \triangleq \int_{-\infty}^{\infty} g(\xi) h(x - \xi) d\xi$$

Autocorrelation: Let $g(x)$ be a function satisfying $\int_{-\infty}^{\infty} |g(x)|^2 dx < \infty$ (finite energy) then

$$\Gamma_g(x) \triangleq (g^* \star g)(x) \triangleq \int_{-\infty}^{\infty} g(\xi) g^*(\xi - x) d\xi = g(x) * g^*(-x)$$

Cross correlation: Let $g(x)$ and $h(x)$ be functions with finite energy. Then

$$(g^* \star h)(x) \triangleq \int_{-\infty}^{\infty} g^*(\xi) h(\xi + x) d\xi = \int_{-\infty}^{\infty} g^*(\xi - x) h(\xi) d\xi = (h^* \star g)^*(-x)$$

The Delta Function: $\delta(x)$

- Scaling: $\delta(ax) = \frac{1}{|a|}\delta(x)$
- Sifting: $\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = f(a)$

  $\int_{-\infty}^{\infty} \delta(x) f(x+a) dx = f(a)$
- Convolution: $\delta(x) * f(x) = f(x)$
- Product: $h(x)\delta(x) = h(0)\delta(x)$
- $\delta^2(x)$ - no meaning
- $\delta(x) * \delta(x) = \delta(x)$
- Fourier Transform of $\delta(x)$: $\mathcal{F}\{\delta(x)\} = 1$
- Derivatives:
  - $\int_{-\infty}^{\infty} \delta^{(n)}(x) f(x) dx = (-1)^n f^{(n)}(0)$
  - $\delta'(x) * f(x) = f'(x)$
  - $x\delta(x) = 0$
  - $x\delta'(x) = -\delta(x)$
- Meaning of $\delta[h(x)]$:

$$\delta[h(x)] = \sum_i \frac{\delta(x - x_i)}{|h'(x_i)|}$$

The Shah Function: $\text{Ш}(x)$

- Sampling: $\text{Ш}(x) g(x) = \sum_{n=-\infty}^{\infty} g(n)\delta(x-n)$
- Replication: $\text{Ш}(x) * g(x) = \sum_{n=-\infty}^{\infty} g(x-n)$
- Fourier Transform: $\mathcal{F}\{\text{Ш}(x)\} = \text{Ш}(s)$
- Scaling: $\text{Ш}(ax) = \sum \delta(ax - n) = \frac{1}{|a|} \sum \delta(x - \frac{n}{a})$

Even and Odd Impulse Pairs

Even: $\text{II}(x) = \frac{1}{2}\delta(x + \frac{1}{2}) + \frac{1}{2}\delta(x - \frac{1}{2})$

Odd: $\text{I}_\text{I}(x) = \frac{1}{2}\delta(x + \frac{1}{2}) - \frac{1}{2}\delta(x - \frac{1}{2})$

Fourier Transforms: $\mathcal{F}\{\text{II}(x)\} = \cos \pi s$

$\mathcal{F}\{\text{I}_\text{I}(x)\} = i \sin \pi s$

Fourier Transform Theorems

- Linearity: $\mathcal{F}\{\alpha f(x) + \beta g(x)\} = \alpha F(s) + \beta G(s)$
- Similarity: $\mathcal{F}\{g(ax)\} = \frac{1}{|a|} G(\frac{s}{a})$
- Shift: $\mathcal{F}\{g(x-a)\} = e^{-i2\pi as} G(s)$

  $\mathcal{F}\{g(ax-b)\} = \frac{1}{|a|} e^{-i2\pi s \frac{b}{a}} G(\frac{s}{a})$
- Rayleigh's: $\int_{-\infty}^{\infty} |g(x)|^2 dx = \int_{-\infty}^{\infty} |G(s)|^2 ds$
- Power: $\int_{-\infty}^{\infty} f(x) g^*(x) dx = \int_{-\infty}^{\infty} F(s) G^*(s) ds$
- Modulation:

$$\mathcal{F}\{g(x)cos(2\pi s_0 x)\} = \frac{1}{2}[G(s - s_0) + G(s + s_0)]$$

- Convolution: $\mathcal{F}\{f * g\} = F(s)G(s)$

- Autocorrelation: $\mathcal{F}\{g^* \star g\} = |G(s)|^2$

- Cross Correlation: $\mathcal{F}\{g^* \star f\} = G^*(s)F(s)$

- Derivative:

  - $\mathcal{F}\{g'(x)\} = i2\pi s G(s)$
  - $\mathcal{F}\{g^{(n)}(x)\} = (i2\pi s)^n G(s)$
  - $\mathcal{F}\{x^n g(x)\} = (\frac{i}{2\pi})^n G^{(n)}(s)$

- Fourier Integral: If $g(x)$ is of bounded variation and is absolutely integrable, then

$$\mathcal{F}^{-1}\{\mathcal{F}\{g(x)\}\} = \frac{1}{2}[g(x^+) + g(x^-)]$$

- Moments:

$$\int_{-\infty}^{\infty} f(x)dx = F(0)$$

$$\int_{-\infty}^{\infty} xf(x)dx = \frac{i}{2\pi}F'(0)$$

$$\int_{-\infty}^{\infty} x^n f(x)dx = (\frac{i}{2\pi})^n F^{(n)}(0)$$

- Miscellaneous:
  If $\mathcal{F}\{g(x)\} = G(s)$ then $\mathcal{F}\{G(x)\} = g(-s)$
  and $\mathcal{F}\{g^*(x)\} = G^*(-s)$

$$\mathcal{F}\left\{\int_{-\infty}^{x} g(\xi)d\xi\right\} = \tfrac{1}{2}G(0)\delta(s) + \frac{G(s)}{i2\pi s}$$

Function Widths

- Equivalent Width

$$W_f \triangleq \frac{\int_{-\infty}^{\infty} f(x)dx}{f(0)} = \frac{F(0)}{f(0)}$$

$$= \frac{F(0)}{\int_{-\infty}^{\infty} F(s)ds} = \frac{1}{W_F}$$

- Autocorrelation Width

$$W_{f^*\star f} \triangleq \frac{\int_{-\infty}^{\infty} f^* \star f \; dx}{f^* \star f \;|_{x=0}}$$

$$= \frac{|F(0)|^2}{\int_{-\infty}^{\infty} |F(s)|^2 ds} = \frac{1}{W_{|F|^2}}$$

- Standard Deviation of Instantaneous Power: Δx

$$(\Delta x)^2 \triangleq \frac{\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx}{\int_{-\infty}^{\infty} |f(x)|^2 dx} - \left[\frac{\int_{-\infty}^{\infty} x|f(x)|^2 dx}{\int_{-\infty}^{\infty} |f(x)|^2 dx}\right]^2$$

$$(\Delta s)^2 \triangleq \frac{\int_{-\infty}^{\infty} s^2 |F(s)|^2 ds}{\int_{-\infty}^{\infty} |F(s)|^2 ds} - \left[\frac{\int_{-\infty}^{\infty} s|F(s)|^2 ds}{\int_{-\infty}^{\infty} |F(s)|^2 ds}\right]^2$$

  - Uncertainty Relation: $(\Delta x)(\Delta s) \geq \frac{1}{4\pi}$

Central Limit Theorem

Given a function $f(x)$, if $F(s)$ has a single absolute maximum at $s = 0$; and, for sufficiently small $|s|$, $F(s) \approx a - cs^2$ where $0 < a < \infty$ and $0 < c < \infty$, then:

$$\lim_{n\to\infty} \frac{[\sqrt{n}f(\sqrt{n}x)]^{*n}}{a^n} = \sqrt{\frac{\pi a}{2}} e^{-\frac{\pi a}{c}x^2}$$

and

$$[f(x)]^{*n} \approx \frac{a^{n+\frac{1}{2}}}{n^{\frac{1}{2}}} \sqrt{\frac{\pi}{c}} e^{-\frac{\pi a}{cn}x^2}$$

Linear Systems

For a linear system $w(t) = \mathcal{S}[v(t)]$ with response $h(t,\tau)$ to a unit impulse at time τ:

$$\mathcal{S}[\alpha v_1(t) + \beta v_2(t)] = \alpha\mathcal{S}[v_1(t)] + \beta\mathcal{S}[v_2(t)]$$

$$w(t) = \int_{-\infty}^{\infty} v(\tau)h(t,\tau)d\tau$$

If such a system is time-invariant, then:

$$w(t-\tau) = \mathcal{S}[v(t-\tau)]$$

and

$$w(t) = \int_{-\infty}^{\infty} v(\tau)h(t-\tau)d\tau = (v * h)(t)$$

The eigenfunctions of any linear time-invariant system are $e^{i2\pi f_0 t}$, since for a system with transfer function H(s), the response to an input of $v(t) = e^{i2\pi f_0 t}$ is given by: $w(t) = H(f_0)e^{i2\pi f_0 t}$.

Sampling Theory

$$\hat{g}(x) = \text{III}(\frac{x}{X})g(x) = X\sum_{n=-\infty}^{\infty} g(nX)\delta(x - nX)$$

$$\hat{G}(s) = X Ш(Xs) * G(s) = \sum_{n=-\infty}^{\infty} G(s - \frac{n}{X})$$

Whittaker-Shannon-Kotelnikov Theorem: For a bandlimited function $g(x)$ with cutoff frequencies $\pm s_c$, and with no discrete sinusoidal components at frequency s_c,

$$g(x) = \sum_{n=-\infty}^{\infty} g(\frac{n}{2s_c}) sinc[2s_c(x - \frac{n}{2s_c})]$$

Fourier Tranforms for Periodic Functions

For a function $p(x)$ with period L, let $f(x) = p(x) \sqcap (\frac{x}{L})$. Then

$$p(x) = f(x) * \sum_{n=-\infty}^{\infty} \delta(x - nL)$$

$$P(s) = \frac{1}{L} \sum_{n=-\infty}^{\infty} F(\frac{n}{L}) \delta(s - \frac{n}{L})$$

The complex fourier series representation:

$$p(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi \frac{n}{L} x}$$

where

$$c_n = \frac{1}{L} F(\frac{n}{L}) = \frac{1}{L} \int_{-L/2}^{L/2} p(x) e^{-i2\pi \frac{n}{L} x} dx$$

The Discrete Fourier Transform

Let $g(x)$ be a physical process, and let $f(x) = g(x)$ for $0 \leq x \leq L$, $f(x) = 0$ otherwise. Suppose f(x) is approximately bandlimited to $\pm B$ Hz, so we sample $f(x)$ every $1/2B$ seconds, obtaining $N = \lfloor 2BL \rfloor$ samples.
The Discrete Fourier Transform:

$$F_m = \sum_{n=0}^{N-1} f_n e^{-i\frac{2\pi mn}{N}} \quad \text{for } m = 0, \ldots, N-1$$

The Inverse Discrete Fourier Transform:

$$f_n = \frac{1}{N} \sum_{m=0}^{N-1} F_m e^{i\frac{2\pi mn}{N}} \quad \text{for } n = 0, \ldots, N-1$$

Convolution:

$$h_n = \sum_{k=0}^{N-1} f_k g_{n-k} \quad \text{for } n = 0, \ldots, N-1 \text{ where } f,\ g \text{ are periodic}$$

Serial Product:

$$h_n = \sum_{k=0}^{N-1} f_k g_{n-k} \quad \text{for } n = 0, \ldots, 2N-2 \text{ where } f,\ g \text{ are not periodic}$$

DFT Theorems

- Linearity: $\mathcal{DFT}\{\alpha f_n + \beta g_n\} = \alpha F_m + \beta G_m$
- Shift: $\mathcal{DFT}\{f_{n-k}\} = F_m e^{-i\frac{2\pi}{N}km}$ (f periodic)
- Parseval's: $\sum_{n=0}^{N-1} f_n g_n^* = \frac{1}{N} \sum_{m=0}^{N-1} F_m G_m^*$
- Convolution: $F_m G_m = \mathcal{DFT}\{\sum_{k=0}^{N-1} f_k g_{n-k}\}$

The Hilbert Transform

The Hilbert Transform of f(x):

$$F_{Hi}(x) \triangleq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x} d\xi \quad \text{(CPV)}$$

The Inverse Hilbert Transform:

$$f(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{F_{Hi}(\xi)}{\xi - x} d\xi + f_{DC} \quad \text{(CPV)}$$

- Impulse response: $-\frac{1}{\pi x}$
- Transfer function: $i\ sgn(s)$
- Causal functions: A causal function $g(x)$ has Fourier Transform $G(s) = R(s) + iI(s)$, where $I(s) = \mathcal{H}\{R(s)\}$.
- Analytic signals: The analytic signal representation of a real-valued function $v(t)$ is given by:

$$z(t) \triangleq \mathcal{F}^{-1}\{2H(s)V(s)\} = v(t) - iv_{Hi}(t)$$

- Narrow Band Signals: $g(t) = A(t)\cos[2\pi f_0 t + \phi(t)]$
  - Analytic approx: $z(t) \approx A(t) e^{i[2\pi f_0 t + \phi(t)]}$
  - Envelope: $A(t) = \mid z(t) \mid$
  - Phase: $\arg[g(t)] = 2\pi f_0 t + \phi(t)$
  - Instantaneous freq: $f_i = f_0 + \frac{1}{2\pi}\frac{d}{dt}\phi(t)$

The Two-Dimensional Fourier Transform

$$F(s_x, s_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-i2\pi(s_x x + s_y y)} dx dy$$

The Inverse Two-Dimensional Fourier Transform:

$$f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(s_x, s_y) e^{i2\pi(s_x x + s_y y)} ds_x ds_y$$

The Hankel Transform (zero order):

$$F(q) = 2\pi \int_0^{\infty} f(r) J_0(2\pi r q) r dr$$

The Inverse Hankel Transform (zero order):

$$f(r) = 2\pi \int_0^\infty F(q) J_0(2\pi r q) q dq$$

Projection-Slice Theorem: The 1-D Fourier transform $P_\theta(s)$ of any projection $p_\theta(x')$ through $g(x,y)$ is identical with the 2-D transform $G(s_x, s_y)$ of $g(x,y)$, evaluated along a slice through the origin in the 2-D frequency domain, the slice being at angle θ to the x-axis. i.e.:

$$P_\theta(s) = G(s\cos\theta, s\sin\theta)$$

Reconstruction by Convolution and Backprojection:

$$\begin{aligned} g(x,y) &= \int_0^\pi \mathcal{F}^{-1}\{\mid s \mid P_\theta(s)\} d\theta \\ &= \int_0^\pi f_\theta(x\cos\theta + y\sin\theta) d\theta \\ \text{where } f_\theta(x') &= (2s_c^2 \mathrm{sinc} 2s_c x' - s_c^2 \mathrm{sinc}^2 s_c x') * p_\theta(x') \end{aligned}$$

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