Noise in MRI

Albert Macovski

This study analyzes the signal-to-noise ratio (SNR) in magnetic resonance imaging. The factors that determine the SNR are derived starting from basic principles. The SNR, for a given object, is shown to be proportional to the voxel volume and the square root of the acquisition time. The noise generated by the body is derived using a cylindrical model and is shown to be proportional to the square of the radius and the square root of the length.

Key words: noise; imaging; SNR.

INTRODUCTION

This study attempts to analyze the signal-to-noise ratio (SNR) for magnetic resonance imaging using a general approach. Since MRI is now a mature field, one may well question the appropriateness of such an analysis at this stage. However, I have been motivated by some papers which, in my view, are misleading if not incorrect when analyzing the SNR.

One such error is that of basing the SNR on the bandwidth (BW) of the received signal. This is fundamentally inaccurate since, in any linear system without aliasing, the noise BW is the smallest or limiting BW of the system. In MRI, this is clearly the integration operation over the acquisition time that takes place at the final Fourier transform stage. The error in using the BW of the received signal is illustrated in Fig. 1. Herein, we see two pulse sequences providing the same resolution and the same imaging time, resulting in the same SNR. However, in Fig. 1b, each line in k-space is traversed twice with twice the gradient amplitude, thus doubling the BW of the received signal. This might be done for better immunity to inhomogeneity or other reasons.

This is indicative of the type of error that can result in conceptual and/or quantitative errors. An extreme case involves spectroscopic imaging with time-varying gradients (1, 2), enabling each point in k-space to be periodically sampled over time. Herein, again, the BW of the received signal represents an erroneous view of the resultant noise.

I hasten to add that formulas using this BW may well provide the correct numerical result. This is simply because, in the typical protocol, the BW of the received signal is governed by the gradient amplitude and image size that can be restructured to represent the voxel size. However, this can give the reader the erroneous notion that, if the receiver signal BW is changed, leaving the resolution and imaging time unchanged, the SNR will be altered.

Analysis

In general, the total received signal, including noise, is given by

$$s(t) = \omega_o N \int_V M(\bar{r}) \cos[\omega_o t + \bar{k}(t) \cdot \bar{r}] d\bar{r} + n(t), \qquad [1]$$

where ω_o is the readout frequency and N is the number of turns in the receiver coil. For convenience, proportionality factors have been ignored. The noise signal, $n(t)$, can be decomposed into $n_i(t)\cos\omega_o t + i n_q(t)\sin\omega_o t$.

The signal is demodulated to baseband using the operation

$$S(t) = 2s(t)\cos\omega_o t * h_l(t) + 2is(t)\sin\omega_o t * h_l(t) \qquad [2]$$

$$= \omega_o N \int_V M(\bar{r}) e^{i\bar{k}(t)\cdot\bar{r}} d\bar{r} + n_i(t) + i n_q(t),$$

where $h_l(t)$ represents a low-pass filter for removal of the $2\omega_o$ components, as is usual in synchronous detectors. The factors of 2 are merely used to avoid carrying factors of 1/2 caused by the product operation. They will not affect the SNR.

To reconstruct, as is usual, we take the Fourier transform of the sequence of $S(t)$ values representing the k-space values. This is equivalent to taking the conjugate phase distribution at each point $\bar{r}_o$, and integrating over time giving

$$\hat{M}(\bar{r}_o) = \frac{1}{T} \int_0^T S(t) e^{-i\bar{k}(t)\cdot\bar{r}_o} dt, \qquad [3]$$

where $\hat{M}(\bar{r}_o)$ is the estimate of the magnetic moment density at each point $\bar{r}_o$, and T is the total acquisition time. Substituting for $S(t)$ and interchanging orders of integration, we have

$$\hat{M}(\bar{r}_o) = \int_V M(\bar{r})\, d\bar{r} \frac{1}{T} \int_0^T e^{i\bar{k}(t)\cdot(\bar{r}-\bar{r}_o)}\, dt + \frac{1}{T} \int_0^T e^{-i\bar{k}(t)\cdot\bar{r}_o} (n_i + i n_q)\, dt. \qquad [4]$$

Since the second integral is the transform of the acquired k-space values, we can make the substitution

$$h(\bar{r} - \bar{r}_o) = \frac{1}{T} \int_0^T e^{i\bar{k}(t)\cdot(\bar{r}-\bar{r}_o)} dt, \qquad [5]$$

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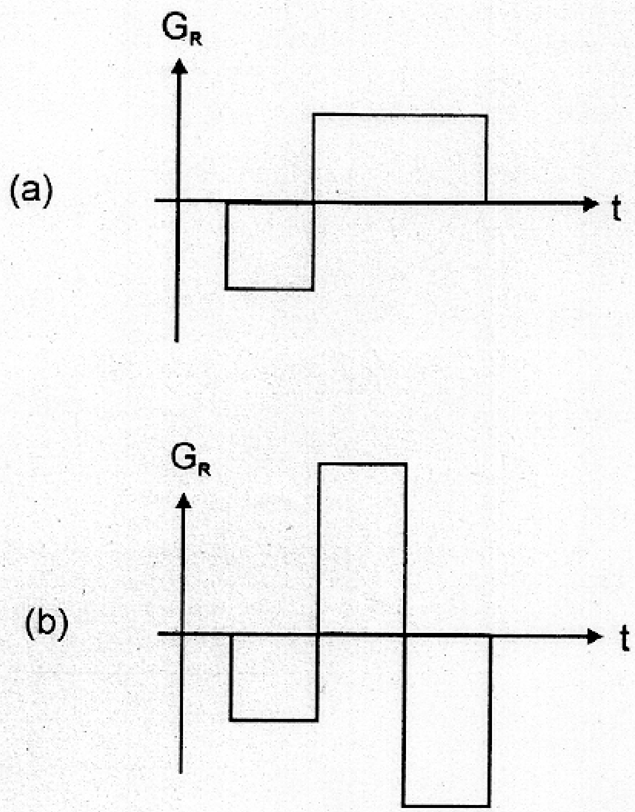

FIG. 1. Gradient readout waveforms producing the same SNR with different received signal BWs.

where h is the system impulse response as long as $d\bar{k}(t)/dt$ is a constant, representing a constant velocity k-space scan, so that we effectively have an integration over $\bar{k}(t)$. Substituting, we have

$$\hat{M}(\bar{r}_o) = \omega_o N \int_V M(\bar{r}) h(\bar{r} - \bar{r}_o)\, d\bar{r} + \frac{1}{T}\int_0^T e^{-i\bar{k}(t)\cdot\bar{r}_o}(n_i + in_q)\, dt. \qquad [6]$$

We decompose our estimate of $\hat{M}(\bar{r}_o)$ into independent signal-and-noise components

$$\hat{M}(\bar{r}_o) = M_{sig} + M_{ns}. \qquad [7]$$

To estimate the signal value, as is normal practice, we assume a uniform $M(r)$ having a value χB_o as a result of an object with static nuclear susceptibility χ in a field B_o. The integration over h, with a constant M, represents the volume of a voxel V_h providing a signal value

$$M_{sig} = \omega_o N \chi B_o V_h. \qquad [8]$$

Substituting the Larmor relationship, $\omega_o = \gamma B_o$, we have

$$M_{sig} = \omega_o^2 N \chi V_h / \gamma. \qquad [9]$$

To estimate the noise variance, we decompose the noise into a real and imaginary part as given by

$$\begin{aligned} M_{ns} &= \frac{1}{T}\int_0^T e^{-i\bar{k}(t)\cdot\bar{r}_o}(n_i + in_q)\, dt \\ &= \frac{1}{T}\int_0^T ([\cos(\bar{k}\cdot\bar{r}_o)n_i + \sin(\bar{k}\cdot\bar{r}_o)n_q] \\ &\quad + i[-\sin(\bar{k}\cdot\bar{r}_o)n_i + \cos(\bar{k}\cdot\bar{r}_o)n_q])\, dt \qquad [10] \\ &= (n_{re} + in_{im}) * h_T, \end{aligned}$$

where the normalized integration over T is represented by convolution with h_T, a rectangular function having a duration of T and unity area.

A linear detector will extract only the real portion of the noise. Often, the signal is plagued by low-frequency phase shifts throughout the image caused by inhomogeneity, susceptibility, flow, etc. As a result, to avoid spurious signals, the magnitude of $M(r)$ is used as the detected signal. In the high SNR case, where the predetection SNR is sufficiently high, ≥ 10, the magnitude detector behaves just like a linear synchronous detector since only the noise in-phase with the signal will contribute, with the quadrature noise rejected. For low values of predetector SNR, the statistics become Rician (3) and ultimately Rayleigh-distributed, providing a much more adverse resultant SNR (4). These problems can be remedied by various approaches to linear detection, such as homodyne detection, a form of synchronous detection, that avoids the nonlinear magnitude operation (5). In any case, we can reasonably assume that we are dealing solely with the real part of the noise as MRI is normally practiced.

The noise variance of each portion of the noise signal is given by

$$\begin{aligned} \sigma_M^2 &= \sigma_{re}^2 = \sigma_{im}^2 \\ &= E\{([\cos(\bar{k}\cdot\bar{r}_o)n_i + \sin(\bar{k}\cdot\bar{r}_o)n_q] * h_T)^2\} \qquad [11] \\ &= \frac{1}{2}[\sigma_i^2 + \sigma_q^2] = \sigma_n^2, \end{aligned}$$

where E, the expected values of the $\cos^2$ and $\sin^2$ terms, are each 1/2, and σ_n^2 is the variance of $n(t)$. We evaluate σ_n^2 as

$$\sigma_n^2 = 2K\mathcal{T}R \int [H_T(f)]^2 df, \qquad [12]$$

Where $\mathcal{T}$ is the absolute temperature, K is Boltzman's constant, R is the effective resistance, and $H_T(f)$ is the frequency domain equivalent of the integration impulse response $h_T = (1/T)\mathrm{rect}(t/T)$ providing $H_T(f) = \mathrm{sinc} fT$.

Using the Power Theorem, we have

$$\int \mathrm{sinc}^2(fT)\, df = \int [(1/T)\mathrm{rect}(t/T)]^2\, dt = 1/T, \qquad [13]$$

providing

$$\sigma_n^2 = 2K\mathcal{T}R/T \qquad [14]$$

and an SNR of

$$\text{SNR} = \frac{\omega_o^2 N\chi V_h/\gamma}{\sqrt{2K\mathcal{T}R/T}}. \qquad [15]$$

This clearly indicates the important proportionalities to the voxel volume and the square root of the acquisition time. However, without studying the factors in the effective resistance, R, we can be misled into believing that the SNR will increase with the square of the field and the number of turns in the pickup coil.

For these reasons, we now estimate the value of the effective resistance, R, using a simple cylindrical geometry. Herein, we first assume that all of the losses are in the object or body, and the coil losses and amplifier noise are negligible. In addition, we assume that the body losses themselves are caused by the induced voltages caused by the time-varying magnetic fields and not by the dielectric losses in the body that might be caused by coil voltages. These latter losses can be minimized through the use of a Faraday shield (6) or by minimizing the coil electromotive force through distributing the tuning capacity throughout the coil.

Body Resistance

To calculate the resultant losses caused by the induced voltages, we make use of reciprocity. We calculate the power dissipated from an applied voltage and use that power to calculate the effective resistance, R. For simplicity, we assume a cylindrical solenoidal coil surrounding the body as shown in Fig. 2. We excite the coil with a unity peak amplitude current $I = \cos\omega_o t$. The resultant average power dissipated by the object or body is given by

$$P_{av} = W = \frac{I_p^2 R}{2} = \frac{R}{2} \qquad [16]$$

where I_p is the peak of the sinusoidal current; unity in this case. Since $R = 2W$, we proceed to find the average power W. As shown in Fig. 3, we divide the body into cylindrical shells, each of length l and thickness dr. The magnetic field in the solenoid, which is assumed uniform, is given by

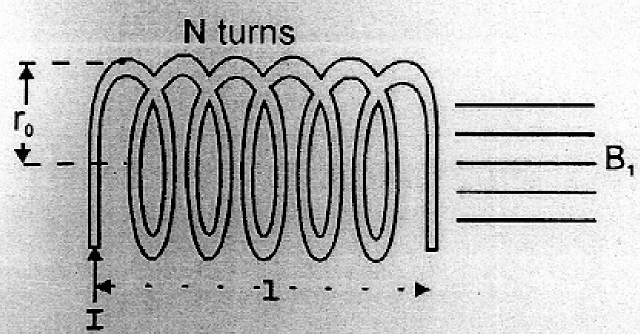

FIG. 2. Cylindrical body coil.

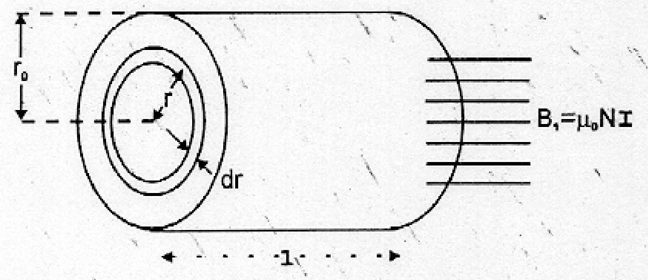

FIG. 3. Calculating body losses using resistive shells.

$$B_1 = \mu_o N I = \mu_o N\cos\omega_o t. \qquad [17]$$

The voltage, V, induced in the cylindrical shell, a single turn coil, is given by

$$V = d\phi/dt = A\frac{dB_1}{dt} = \pi r^2\omega_o\mu_o N\sin\omega_o t, \qquad [18]$$

where $A = \pi r^2$ is the area subtended by the cylindrical shell. The peak-induced voltage, V_p, is used to calculate the average power dissipated in the shell as given by

$$dW = \frac{V_p^2}{2}\,dG, \qquad [19]$$

where the differential conductance of the cylindrical shell, dG, is given by the cross-sectional area of the single turn divided by the product of the length and the resistivity ρ as given by

$$dG = \frac{l\,dr}{2\pi r\rho}. \qquad [20]$$

We integrate dW over the entire cylinder to find the total dissipated power

$$W = \int_0^{r_o} dW(r) = \int_0^{r_o} \frac{(\pi r^2\omega_o\mu_o N)^2}{2}\,\frac{l\,dr}{2\pi r\rho}. \qquad [21]$$

The effective resistance, R, is then given by

$$R = 2W = \frac{\pi\omega_o^2\mu_o^2 N^2 l r_o^4}{8\rho}, \qquad [22]$$

resulting in a noise standard deviation

$$\sigma_n = \frac{\omega_o\mu_o N r_o^2}{2}\sqrt{\frac{K\mathcal{T}\pi l}{\rho T}} \qquad [23]$$

Our calculated SNR is, therefore,

$$\text{SNR} = \frac{M_s}{\sigma_n} = \frac{\omega_o^2 N\chi V_h/\gamma}{\dfrac{\omega_o\mu_o N r_o^2}{2}\sqrt{\dfrac{K\mathcal{T}\pi l}{\rho T}}}$$

$$= \left[\frac{2\chi\sqrt{\rho}}{\gamma\mu_o\sqrt{K\mathcal{T}\pi}}\right]\left[\frac{1}{r_o^2\sqrt{l}}\right][\omega_o V_h\sqrt{T}] \qquad [24]$$

$$= Cf(Ob)f(\text{Im}),$$

where the first bracketed term represents the physical constants, C, that are beyond our control; the second bracketed term represents the object dimensions, $f(Ob)$; and the third bracketed term represents the chosen imaging parameters of frequency (magnetic field), voxel size, and imaging time $f(Im)$. Note that the SNR is independent of the number of turns, N. This is only true where the noise is dominated by body losses, the normal case at the higher magnetic fields.

The second term, dependent on object dimensions, will vary with the size and shape of the object. For example, using a spherical object, the SNR varies as $V^{2/3}$, where V is the volume of the object (7). In the cylindrical case, for a given length, the SNR varies inversely with the cross-sectional area.

Of course, many other factors can influence the SNR in a predictable manner, such as relaxation, small tip angles, voxel inhomogeneity, etc. However, the important message is that the SNR is proportional to the voxel size and the square root of the acquisition time, independent of how these are arrived at and independent of the BW of the received signal.

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