SLR Pulse Design

The following pages contain excerpts from the lecture notes for "RF Pulse Design for Magnetic Resonance Imaging," a course taught at Stanford University by John M. Pauly.

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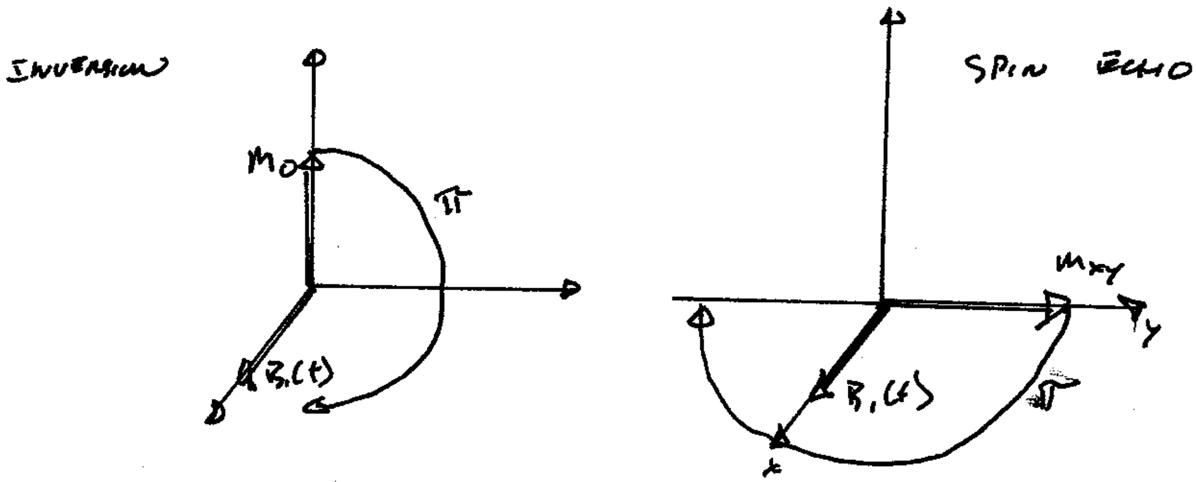
LARGE TIP ANGLE PULSES

HOW DO WE DESIGN INVERSION OR SPIN-ECHO PULSES?



SMALL-TIP-ANGLE APPROXIMATION

$$m_{xy}(\underline{r}, t) = i m_0 \int_{-\infty}^{t} \gamma B_1(\tau)\, e^{i 2\pi \underline{k}(\tau, t)\cdot \underline{r}}\, d\tau$$

ASSUMES INITIAL $\underline{M} = (0,0,1) m_0$, SMALL ANGLE (WORKS PRETTY WELL TO $\pi/2$, AS WE'LL SEE)

FOR 180, DESIGN SMALL-TIP-ANGLE PULSE

$\Rightarrow$ SCALE TO 180

DOES NOT WORK SO WELL

WE REALLY NEED TO DEAL DIRECTLY WITH ROTATIONS

MOTION OF THE MAGNETIZATION GOVERNED BY BLOCH EQUATION

$$\frac{d}{dt}\begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} = \begin{pmatrix} 0 & \gamma G x & -\gamma B_{1,y} \\ -\gamma G x & 0 & \gamma B_{1,x} \\ \gamma B_{1,y} & -\gamma B_{1,x} & 0 \end{pmatrix}\begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix}$$

NEGLECTING T_1, T_2 AND ASSUME WE ARE EXACTLY ON RESONANCE

WE CAN WRITE THIS MORE COMPACTLY USING THE SPIN MATRICES

$$S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$S_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$S_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{S} = (S_x, S_y, S_z)$$ VECTOR OF MATRICES

THEN

$$\frac{d}{dt}\begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} = \Bigg[-\gamma B_{1,x} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$- \gamma B_{1,y} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} - \gamma Gx \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Bigg] \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix}$$

$$\frac{d}{dt}\underline{m} = \left[\left(-\gamma B_{1x}, -\gamma B_{1,y}, -\gamma Gx \right) \cdot \underline{S} \right] \underline{m}$$

DEFINE

$$\omega = -\gamma \sqrt{(B_{1,x})^2 + (B_{1,y})^2 + (Gx)^2}$$ RATE OF ROTATION

$$\underline{n} = \frac{\gamma}{|\omega|} \left(B_{1,x}, B_{1,y}, Gx \right)$$ AXIS OF ROTATION

WE WILL ASSUME (FOR NOW)

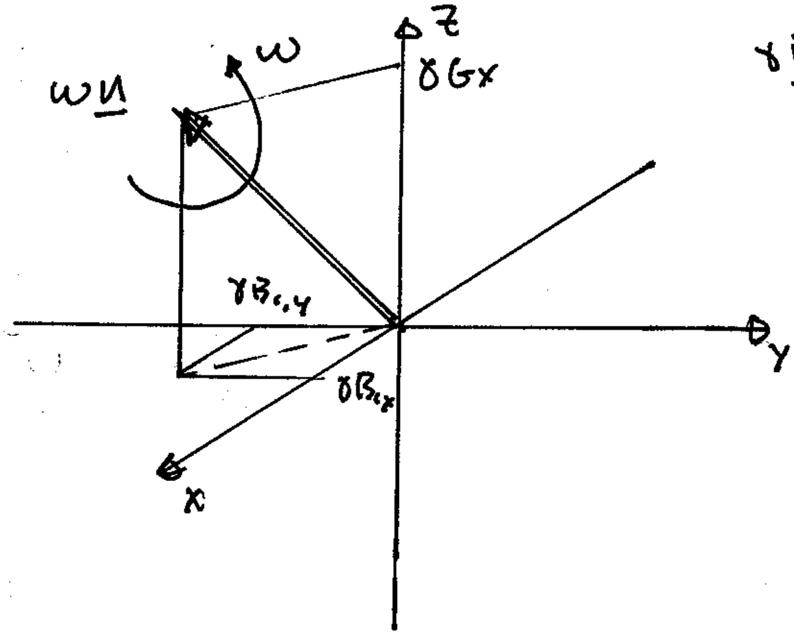
G IS CONSTANT

$B_{1,x}$ AND $B_{1,y}$ ARE TIME VARYING

HENCE

$\omega, \underline{n}$ ARE TIME VARYING

THE $(-\gamma)$ IS DUE TO PROTONS PRECESSING IN THE LEFT-HAND SENSE, AND WE WILL USE RIGHT-HANDED ROTATIONS.



$$\gamma \underline{B} = \gamma(B_{1,x}, B_{1,y}, Gx) = -\omega \underline{n}$$

MAGNETIZATION ROTATES ABOUT $\underline{n}$ AT A RATE ω

THEN

$$\frac{d}{dt}\underline{m} = \omega(\underline{n} \cdot \underline{S})\underline{m}$$

SOLUTION WILL BE OF THE FORM

$$\underline{m}(T) = R\,\underline{m}(0)$$

WHERE R IS A 3×3 ORTHONORMAL MATRIX

NOT OBVIOUS HOW TO FIND R

(WE'LL SEE A FEW SPECIAL CASES LATER...)

NOTE THAT

$$R = e^{\int_{-\infty}^{t} \omega(\tau)(\underline{n}(\tau)\cdot\underline{S})\,d\tau}$$

IS NOT IN GENERAL A SOLUTION. THIS IS BECAUSE

$$e^{A+B} \neq e^{A} e^{B}$$

UNLESS A AND B COMMUTE $(AB = BA)$

ONE CASE WHERE IT DOES HOLD IS $\underline{n}$ CONSTANT.

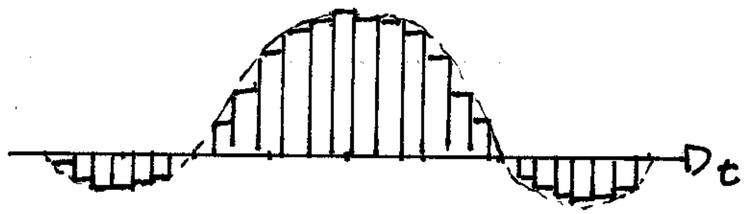
$$R = e^{(\underline{n}\cdot\underline{S})\int_{-\infty}^{t} \omega(\tau)\,d\tau}$$

THIS WILL BE AN IMPORTANT SPECIAL CASE LATER.

NOTE THAT e^{A}, WHERE A IS A MATRIX, IS

$$e^{A} = I + A + \tfrac{1}{2}A^{2} + \ldots$$

PIECEWISE CONSTANT APPROXIMATION



RF MADE UP OF SHORT RECTANGLES, EACH PRODUCING A FLIP ANGLE

$$\gamma B_1(t_i)\Delta t$$

WHERE Δt IS THE WIDTH OF THE RECTANGLE

THE ROTATION PRODUCED IS

$$R_i = e^{(\underline{n}_i \cdot \underline{S})\omega_i \Delta t}$$

WHICH WE CAN SOLVE FOR AS A 3x3 ORTHONORMAL MATRIX

THE TOTAL ROTATION IS THEN

$$R = R_n R_{n-1} \cdots R_2 R_1$$

EXAMPLE: LET $\underline{n} = (1,0,0)$, A ROTATION ABOUT X, BY AN ANGLE θ

$$R = e^{S_x \theta}$$

$$= I + (\theta S_x) + \frac{1}{2}(\theta S_x)^2 + \frac{1}{6}(\theta S_x)^3$$

AFTER SOME TEDIOUS CALCULATION

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

GENERAL SOLUTION ARBITRARY $\underline{n}$, θ

$$R = e^{(\underline{n}\cdot\underline{S})\theta}$$

$$= I\cos\theta + (n^T n)(1-\cos\theta) + (\underline{n}\cdot\underline{S})\sin\theta$$

THIS IS THE SO(3) REPRESENTATION OF ROTATIONS

3×3 ORTHONORMAL MATRICES

GENERALLY TOO CUMBERSOME TO DO BY HAND!

SIMPLER REPRESENTATION

WE CAN ALSO REPRESENT ROTATIONS USING THE 2x2 UNITARY MATRICES

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

LET

$$\underline{\sigma} = (\sigma_x , \sigma_y , \sigma_z)$$

THE DIFFERENTIAL EQUATION THAT CORRESPONDS TO THE BLOCH EQUATION IS

$$\dot{\psi} = \frac{i\omega}{2} (\underline{n} \cdot \underline{\sigma}) \psi$$

WHERE $\underline{n}$ AND ω ARE THE SAME AS BEFORE

$$\omega = -\gamma \sqrt{B_{1,x}^2 + B_{1,y}^2 + (Gx)^2}$$

$$\underline{n} = \frac{\gamma}{|\omega|} (B_{1,x} , B_{1,y} , Gx)$$

ψ IS A SPINOR, WHICH WE WILL COME BACK TO.

FOR THE CASE WHERE $\underline{n}$ AND ω ARE CONSTANT (ONE SAMPLE OF PIECE-WISE CONSTANT PULSE)

DEFINE

$$\theta = \omega \Delta t$$

SOLUTION IS

$$\psi_{i+1} = Q\,\psi_i$$

WHERE

$$Q = e^{i\frac{\theta}{2}(\underline{n}\cdot\underline{\sigma})}$$

$$= I\cos\frac{\theta}{2} - i(\underline{n}\cdot\underline{\sigma})\sin\frac{\theta}{2}$$

SO

$$Q = \overbrace{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}}^{I}\cos\theta/2 + \overbrace{\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}}^{\sigma_x}\left(-in_x\sin\frac{\theta}{2}\right)$$

$$+ \underbrace{\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}}_{\sigma_y}\left(-in_y\sin\theta/2\right) + \underbrace{\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}}_{\sigma_z}\left(-in_z\sin\theta/2\right)$$

$$= \begin{pmatrix}\cos\theta/2 - in_z\sin\theta/2 & -i(n_x - in_y)\sin\theta/2\\ -i(n_x + in_y)\sin\theta/2 & \cos\theta/2 + in_z\sin\theta/2\end{pmatrix}$$

DEFINE

$$\boxed{\begin{aligned}\alpha &= \cos\theta/2 - in_z\sin\theta/2\\ \beta &= -i(n_x + in_y)\sin\theta/2\end{aligned}}$$

THEN

$$\boxed{Q = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}}$$

TWO COMPLEX NUMBERS DETERMINE ROTATION!

α AND β ARE THE CAYLEY-KLEIN PARAMETERS

ONE ADDITIONAL CONSTRAINT IS

$$\alpha\alpha^* + \beta\beta^* = 1$$

HENCE, THERE ARE ONLY 3 FREE PARAMETERS.

FOR OUR RF PULSE, THE TOTAL ROTATION IS

$$Q = Q_n Q_{n-1} \cdots Q_2 Q_1$$

PRODUCT OF 2x2 UNITARY MATRICES

SU(2) REPRESENTATION OF ROTATIONS.

HOWEVER, IT IS ACTUALLY EVEN SIMPLER.

LET

$$Q_n = \begin{pmatrix} a_n & -b_n^* \\ b_n & a_n^* \end{pmatrix}$$

BE ONE OF THE INCREMENTAL ROTATIONS, AND

$$\begin{pmatrix} \alpha_n & -\beta_n^* \\ \beta_n & \alpha_n^* \end{pmatrix} = \prod_{j=1}^{n} \begin{pmatrix} a_j & -b_j^* \\ b_j & a_j^* \end{pmatrix}$$

THEN

$$\begin{pmatrix} \alpha_n & -\beta_n^* \\ \beta_n & \alpha_n^* \end{pmatrix} = \begin{pmatrix} a_n & -b_n^* \\ b_n & a_n^* \end{pmatrix} \cdots \underbrace{\begin{pmatrix} a_j & -b_j^* \\ b_j & a_j^* \end{pmatrix} \cdots \begin{pmatrix} a_1 & -b_1^* \\ b_1 & a_1^* \end{pmatrix}}_{\begin{pmatrix} \alpha_j & -\beta_j^* \\ \beta_j & \alpha_j^* \end{pmatrix}}$$

WE REALLY ONLY NEED TO KEEP TRACK OF $(\alpha_j \;\; \beta_j)^T$

$$\begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} = \begin{pmatrix} a_j & -b_j^* \\ b_j & a_j^* \end{pmatrix} \begin{pmatrix} \alpha_{j-1} \\ \beta_{j-1} \end{pmatrix}$$

PROPOGATE α, β BY 2X2 VECTOR MATRIX PRODUCTS.

THE VECTOR $(\alpha_i \ \beta_i)^T$ IS A SPINOR

$$\psi_i = \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}$$

THE INITIAL CONDITION IS NO ROTATION $(\theta=0)$
SO

$$\psi_0 = \begin{pmatrix} \cos\theta/2 - i n_z \sin\theta/2 \\ -i(n_x+in_y)\sin\theta/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

HALF ANGLES

THE SPINOR ROTATIONS ARE ALL HALF ANGLES!

A ROTATION BY 2π GIVES

$$\psi(2\pi) = \begin{pmatrix} \cos 2\pi/2 - i n_z \sin 2\pi/2 \\ -i(n_x+in_y)\sin 2\pi/2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

ABOUT ANY $\underline{n}$. THE SPINOR CHANGES SIGN.

A ROTATION BY 4π GIVES

$$\psi(4\pi) = \begin{pmatrix} \cos(4\pi/2) - i n_z \sin 4\pi/2 \\ -i(n_x+in_y)\sin 4\pi/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

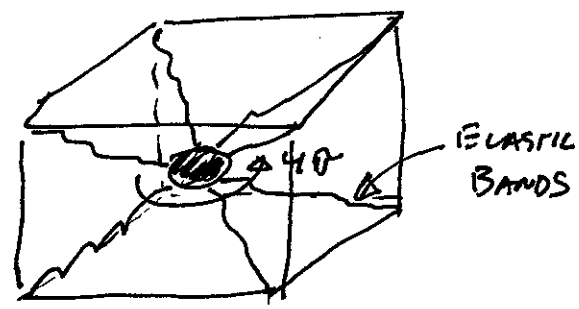
SO 720° IS THE IDENTITY ROTATION.

WHERE DOES THIS COME FROM?

PHYSICS: $\frac{1}{2}$ INTEGER SPIN PARTICLES (FERMIONS)
WAVE FUNCTION CHANGE SIGN WITH 2π ROTATION

EXTENDED OBJECT ROTATION:

OBJECT CONNECTED TO A FRAME BY ELASTIC BANDS



A 2π ROTATION CANNOT BE UNTANGLED

A 4π ROTATION CAN!

IMPLICATIONS FOR PULSE DESIGN

MOST PULSES ARE BETWEEN 0 AND π

THEN

$\cos \theta/2$ GOES FROM 1 TO 0

$\sin \theta/2$ GOES FROM 0 TO 1

COMPLETELY UNAMBIGUOUS! NO PHASE UNWRAPPING PROBLEMS

VERY CONVENIENT

SPIN DOMAIN REPRESENTATION OF ROTATION

ROTATIONS CAN BE REPRESENTED BY MULTIPLICATION OF 2x2 UNITARY MATRICES

$$Q = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}$$

WHERE

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \cos\theta/2 - i n_z \sin\theta/2 \\ -i(n_x + i n_y)\sin\theta/2 \end{pmatrix}$$

AND

$$\underline{n} = (n_x, n_y, n_z)$$

IS THE AXIS OF ROTATION, AND θ IS THE ANGLE.

NOTE THAT:

$$\alpha\alpha^* + \beta\beta^* = 1$$

THE RESULT OF A SEQUENCE OF ROTATIONS $Q_1 \ldots Q_n$ IS

$$Q = Q_n Q_{n-1} \cdots Q_3 Q_2 Q_1$$

CHANGING DOMAINS

MAGNETIZATION TO SPIN DOMAIN

$$\omega = -\gamma \sqrt{B_{1,x}^2 + B_{1,y}^2 + (Gx)^2}$$

$$\underline{n} = \frac{\gamma}{|\omega|}\left(B_{1,x}, B_{1,y}, Gx\right)$$

$$\Theta = \omega \Delta t$$

THEN $(\underline{n}, \Theta)$ DETERMINE (α, β).

SPIN DOMAIN TO MAGNETIZATION

FOR A GIVEN SPINOR $\psi = (\alpha, \beta)^T$, THE (OBSERVABLE) MAGNETIZATION COMPONENTS ARE

$$m_x = \psi^* \sigma_x \psi \qquad m_y = \psi^* \sigma_y \psi \qquad m_z = \psi^* \sigma_z \psi$$

WHERE

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

AND

$$\psi = Q\psi_0 = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix}$$

IF THE MAGNETIZATION IS INITIALLY ALONG +Z AXIS, $\Theta=0$ AND

$$\psi_0 = \begin{pmatrix} \cos\frac{\Theta}{2} - i n_z \sin\Theta/2 \\ -i(n_x + i n_y)\sin\Theta/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

THEN

$$\psi = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

WE CAN THEN COMPUTE

$$m_x = (\alpha^* \ \beta^*)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= (\alpha^* \ \beta^*)\begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

$$= \underline{\alpha^*\beta + \beta^*\alpha}$$

$$m_y = (\alpha^* \ \beta^*)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= (\alpha^* \ \beta^*)\begin{pmatrix} -i\beta \\ i\alpha \end{pmatrix}$$

$$= \underline{-i\alpha^*\beta + i\alpha\beta^*}$$

$$m_z = (\alpha^* \;\; \beta^*)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= (\alpha^* \;\; \beta^*)\begin{pmatrix} \alpha \\ -\beta \end{pmatrix}$$

$$= \underline{\alpha\alpha^* - \beta\beta^*}$$

RECALL THAT

$$\alpha\alpha^* + \beta\beta^* = 1$$

SO

$$m_z = (1 - \beta\beta^*) - \beta\beta^*$$

$$= \underline{\underline{1 - 2\beta\beta^*}}$$

AS USUAL, WE WILL BE INTERESTED IN

$$m_{xy} = m_x + i\, m_y$$

SUBSTITUTING FOR m_x AND m_y

$$m_{xy} = (\alpha^*\beta + \beta^*\alpha) + i(-i\alpha^*\beta + i\alpha\beta^*)$$

$$= \alpha^*\beta + \cancel{\beta^*\alpha} + \alpha^*\beta - \cancel{\alpha\beta^*}$$

$$= \underline{\underline{2\alpha^*\beta}}$$

WE COULD HAVE OBTAINED THIS DIRECTLY BY DEFINING

$$\sigma_{xy} = \sigma_x + i\,\sigma_y$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= 2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$ "RAISING OPERATOR"

THEN

$$m_{xy} = \psi^{*} \sigma_{xy} \psi$$

$$= (\alpha^{*} \ \beta^{*}) \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= (\alpha^{*} \ \beta^{*}) \begin{pmatrix} 2\beta \\ 0 \end{pmatrix}$$

$$= \underline{\underline{2\alpha^{*}\beta}}$$

WE CAN COMPUTE THE VARIOUS TERMS FOR m_{xy} AND m_{xy}^*, AND COLLECT THEM IN A MATRIX

$$\begin{pmatrix} m_{xy}^+ \\ m_{xy}^{+*} \\ m_z^+ \end{pmatrix} = \begin{pmatrix} (\alpha^*)^2 & -\beta^2 & 2\alpha^*\beta \\ -(\beta^*)^2 & \alpha^2 & 2\alpha\beta^* \\ -\alpha^*\beta^* & -\alpha\beta & \alpha\alpha^* - \beta\beta^* \end{pmatrix} \begin{pmatrix} m_{xy}^- \\ m_{xy}^{-*} \\ m_z^- \end{pmatrix}$$

FOR ANY INITIAL MAGNETIZATION $\underline{m}^-$ THERE IS A SIMPLE EXPRESSION FOR $\underline{m}^+$

IMPORTANT SPECIAL CASES

EXCITATION PROFILE, $\underline{m}^- = (0, 0, m_0)$

$$\underline{\underline{m_{xy}^+ = 2\alpha^*\beta\, m_0}}$$

INVERSION / SATURATION PROFILE

$$\underline{\underline{m_z^+ = (\alpha\alpha^* - \beta\beta^*)\, m_0}}$$

LAST TIME

ROTATIONS REPRESENTED BY 2X2 UNITARY MTX'S

$$Q = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} \qquad \psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

WHERE

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \cos\theta/2 - i n_z \sin\theta/2 \\ -i(n_x + i n_y)\sin\theta/2 \end{pmatrix}$$

AND $\underline{n}$ IS THE ROTATION AXIS, AND θ IS ANGLE.

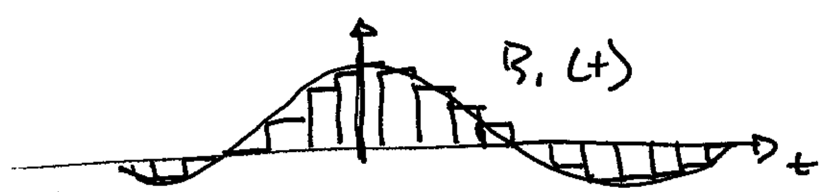
ADDITIONAL CONSTRAINT

$$\alpha\alpha^* + \beta\beta^* = 1$$

SEQUENCE OF ROTATIONS MULTIPLY MTX'S

$$Q = Q_n Q_{n-1} \cdots Q_2 Q_1$$

FOR A RECTANGULAR APPROXIMATION TO CONTINUOUS PULSE



EACH SUBPULSE PRODUCES

$$\omega = -\gamma\sqrt{B_{1,x}^2 + B_{1,y}^2 + (Gx)^2} \qquad \text{FREQUENCY}$$

$$\underline{n} = \frac{\gamma}{|\omega|}\left(B_{1,x},\ B_{1,y},\ Gx\right) \qquad \text{AXIS}$$

$$\Theta = \omega\Delta t \qquad \text{ANGLE}$$

GIVEN ψ, WHAT IS m_x, m_y AND m_z

$$m_x = \psi^* \sigma_x \psi \qquad m_y = \psi^* \sigma_y \psi \qquad m_z = \psi^* \sigma_z \psi$$

WHERE

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

FOR ANY INITIAL m_{xy}^-, m_z^- WE CAN COMPUTE m_{xy}^+, m_z^+

$$\begin{pmatrix} m_{xy}^+ \\ m_{xy}^{+*} \\ m_z^+ \end{pmatrix} = \begin{pmatrix} (\alpha^*)^2 & -\beta^2 & 2\alpha^*\beta \\ -(\beta^*)^2 & \alpha^2 & 2\alpha\beta^* \\ -\alpha^*\beta^* & -\alpha\beta & \alpha\alpha^* - \beta\beta^* \end{pmatrix} \begin{pmatrix} m_{xy}^- \\ m_{xy}^{-*} \\ m_z^- \end{pmatrix}$$

VERY SIMPLE.

IMPORTANT SPECIAL CASES

EXCITATION PROFILE $\underline{m}^- = (0, 0, m_0)$

$$\underline{m_{xy}^+ = 2\alpha^*\beta\, m_0}$$

INVERSION PROFILE $\underline{m}^- = (0, 0, m_0)$

$$m_z^+ = (\alpha\alpha^* - \beta\beta^*)\, m_0$$

$$\underline{m_z^+ = (1 - 2\beta\beta^*)\, m_0}$$

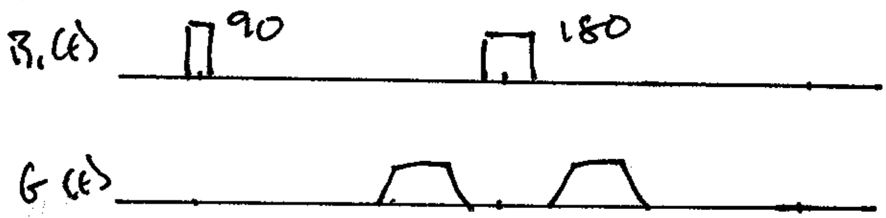
SPIN-ECHO PROFILE, $\underline{m}^- = (m_{xy}, m_{xy}^*, 0)$

$$\underline{\underline{m_{xy}^+ = (\alpha^*)^2 m_{xy}^- - \beta^2 m_{xy}^{-*}}}$$

IF THE INITIAL MAGNETIZATION IS ALONG +Y (FOLLOWING A -90_x), $m_{xy} = i\, m_0$ AND

$$m_{xy}^+ = i\left((\alpha^*)^2 + \beta^2\right) m_0$$

TO IDENTIFY THE TWO TERMS, IT IS USEFUL TO CONSIDER THE FOLLOWING PULSE SEQUENCE



"CRUSHERS"

THE PHASE PRODUCED BY THE GRADIENT IS

$$\phi(x) = \left[-\gamma \int_{\text{LOBE}} G(\tau)\, d\tau\right] \cdot x$$

THE ROTATION IT PRODUCES IS

$$Q_c = \begin{pmatrix} e^{-i\phi(x)/2} & 0 \\ 0 & e^{i\phi(x)/2} \end{pmatrix}$$

THE ROTATION PRODUCED BY 180-CRUSHER COMPLEX:

$$Q_c Q_{180} Q_c = \begin{pmatrix} e^{-i\phi(x)/2} & 0 \\ 0 & e^{i\phi(x)/2} \end{pmatrix} \begin{pmatrix} \alpha_{180} & -\beta_{180}^* \\ \beta_{180} & \alpha_{180}^* \end{pmatrix} \begin{pmatrix} e^{-i\phi(x)/2} & 0 \\ 0 & e^{i\phi(x)/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\phi(x)/2} & 0 \\ 0 & e^{i\phi(x)/2} \end{pmatrix} \begin{pmatrix} \alpha_{180} e^{-i\phi(x)/2} & -\beta_{180}^* e^{i\phi(x)/2} \\ \beta_{180} e^{-i\phi(x)/2} & \alpha_{180}^* e^{i\phi(x)/2} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_{180} e^{-i\phi(x)} & -\beta_{180}^* \\ \beta_{180} & \alpha_{180}^* e^{i\phi(x)} \end{pmatrix}$$

SPIN ECHO IS THEN

$$m_{xy}^+ = (\alpha^*)^2 m_{xy}^- - \beta^2 m_{xy}^{-*}$$

$$= ((\alpha_{180} e^{-i\phi(x)})^*)^2 m_{xy}^- - \beta_{180}^2 m_{xy}^{-*}$$

$$= \underbrace{(\alpha_{180}^*)^2 e^{i2\phi(x)} m_{xy}^-}_{\text{CRUSHED TERM}} - \underbrace{\beta_{180}^2 m_{xy}^{-*}}_{\text{REFOCUSED TERM}}$$

CRUSHED TERM NOT REFOCUSED ($m_{xy}^+ \sim m_{xy}^-$)

REFOCUSED TERM IS SPIN ECHO

NO $\phi(x)$ DEPENDENCE

$m_{xy}^+ \sim (m_{xy}^-)^*$

HENCE

$$\underline{\underline{m^{+}_{xy,CR} = (\alpha^{*})^{2}\, m^{-}_{xy}}} \qquad \text{(STRAIGHT THROUGH)}$$

AND

$$\underline{\underline{m^{+}_{xy,SE} = -\beta^{2}\, m^{-\,*}_{xy}}} \qquad \text{(SPIN-ECHO)}$$

WHERE $(\alpha, \beta)^{T}$ ARE $(\alpha_{180}, \beta_{180})^{T}$

EXAMPLE ROTATION MATRICES

ROTATION ABOUT X

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \cos\theta/2 - i n_z \sin\theta/2 \\ -i(n_x + i n_y)\sin\theta/2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta/2 \\ -i\sin\theta/2 \end{pmatrix}$$

$$Q = \begin{pmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

$$Q(90_x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

$$Q(180_x) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$Q(360_x) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

ROTATIONS ABOUT Y

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \cos\theta/2 - i n_z \sin\theta/2 \\ -i(n_x + i n_y)\sin\theta/2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}$$

$$Q = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

$$Q(90_Y) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$Q(180_Y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

ROTATIONS ABOUT Z

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \cos\theta/2 - i n_z \sin\theta/2 \\ -i(n_x + i n_y)\sin\theta/2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta/2 - i\sin\theta/2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\theta/2} \\ 0 \end{pmatrix}$$

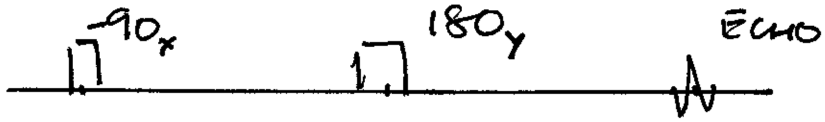
$$Q = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

$$= \begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix}$$

WHERE

$$z = e^{+i\theta}$$

SIMPLE PULSE SEQUENCE EXAMPLE



$$Q(-90_x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

$$Q(180_y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$Q(FP) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$ FREE PRECESSION

$$= \begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix}$$

AT THE ECHO

$$Q = Q(FP)\,Q(180_y)\,Q(FP)\,Q(-90_x)$$

$$\psi = \begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z^{-1/2} \\ i z^{1/2} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix} \begin{pmatrix} -i z^{1/2} \\ z^{-1/2} \end{pmatrix}$$

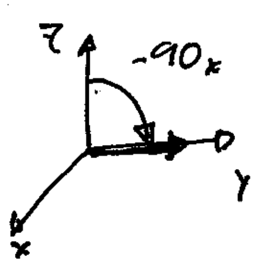
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

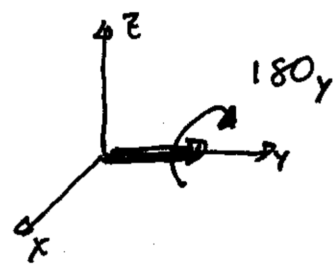
$$m_{xy} = 2 \alpha^* \beta \, m_0$$

$$= 2 \left(\frac{-i}{\sqrt{2}} \right)^* \left(\frac{1}{\sqrt{2}} \right)$$

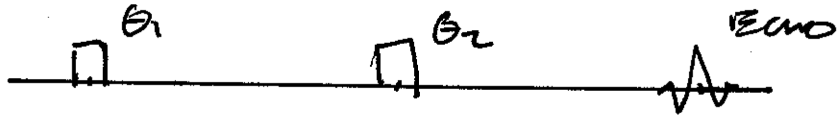
$$= i \, m_0$$

WHICH IS WHAT WE WOULD EXPECT.





MORE INVOLVED EXAMPLE



WHAT IS THE TRANSVERSE MAGNETIZATION AFTER ANY TWO PULSES?

$$Q_1 = \begin{pmatrix} \alpha_1 & -\beta_1^* \\ \beta_1 & \alpha_1^* \end{pmatrix} \qquad Q_2 = \begin{pmatrix} \alpha_2 & -\beta_2^* \\ \beta_2 & \alpha_2^* \end{pmatrix}$$

THEN

$$\psi = \begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix}\begin{pmatrix} \alpha_2 & -\beta_2^* \\ \beta_2 & \alpha_2^* \end{pmatrix}\begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix}\begin{pmatrix} \alpha_1 & -\beta_1^* \\ \beta_1 & \alpha_1^* \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix}\begin{pmatrix} \alpha_2 & -\beta_2^* \\ \beta_2 & \alpha_2^* \end{pmatrix}\begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$$

$$= \begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix}\begin{pmatrix} \alpha_2 & -\beta_2^* \\ \beta_2 & \alpha_2^* \end{pmatrix}\begin{pmatrix} \alpha_1 z^{-1/2} \\ \beta_1 z^{1/2} \end{pmatrix}$$

$$= \begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix}\begin{pmatrix} \alpha_1 \alpha_2 z^{-1/2} - \beta_1 \beta_2^* z^{1/2} \\ \alpha_1 \beta_2 z^{-1/2} + \beta_1 \alpha_2^* z^{1/2} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1 \alpha_2 z^{-1} - \beta_1 \beta_2^* \\ \alpha_1 \beta_2 + \beta_1 \alpha_2^* z^{+1} \end{pmatrix}$$

$$m_{xy}^* = 2\alpha^*\beta$$

$$= 2\left(\alpha_1\alpha_2 z^{-1} - \beta_1\beta_2^*\right)^*\left(\alpha_1\beta_2 + \beta_1\alpha_2^* z^{+1}\right)$$

$$= 2\left(\alpha_1^*\alpha_2^* z - \beta_1^*\beta_2\right)\left(\alpha_1\beta_2 + \beta_1\alpha_2^* z^{+1}\right)$$

$$= 2\alpha_1^*\beta_1\left(\alpha_2^*\right)^2 z^2 + 2\left(\alpha_1^*\alpha_1\alpha_2^*\beta_2 - \beta_1^*\beta_1\alpha_2^*\beta_2\right)z$$

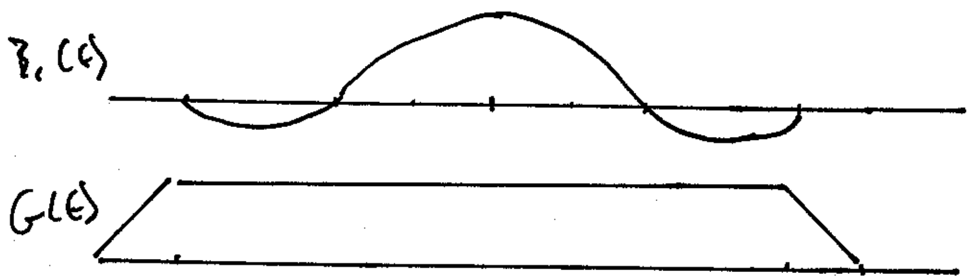
$$- 2\alpha_1\beta_1^*\left(\beta_2\right)^2$$

$$= \overbrace{\underbrace{\left(2\alpha_1^*\beta_1\right)}_{m_{xy,1}}\underbrace{\left(\alpha_2^*\right)^2}_{m_{xy,2,CR}}}^{\text{STRAIGHT THROUGH}} z^2 + \overbrace{\underbrace{\left(\alpha_1^*\alpha_1 - \beta_1^*\beta_1\right)}_{m_{z,1}}\underbrace{\left(2\alpha_2^*\beta\right)}_{m_{xy,2}}}^{\text{PARASITIC}} z$$

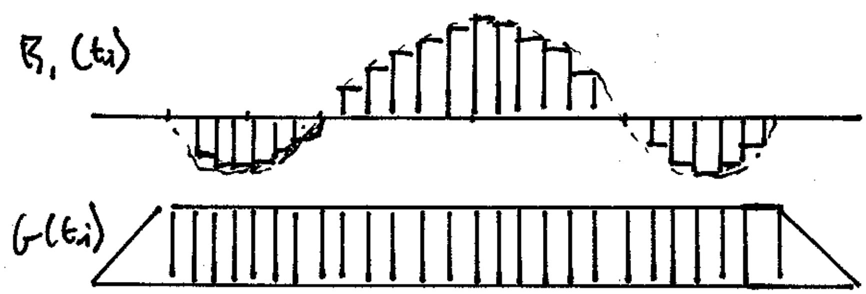
$$- \underbrace{\underbrace{\left(2\alpha_1\beta_1^*\right)}_{m_{xy,1}^*}\underbrace{\left(\beta_2\right)^2}_{m_{xy,2,SE}}}_{\text{SPIN ECHO}}$$

CALCULATING RESPONSE OF LARGE-TIP-ANGLE PULSES

CONTINUOUS, LARGE-TIP-ANGLE PULSE



MODEL AS DISCRETE RECTANGLES



THE i^{th} RECTANGLE PRODUCES A ROTATION

$$\theta_i = -\gamma \Delta t \sqrt{|B_1(t_i)|^2 + (Gx)^2}$$

ABOUT AN AXIS

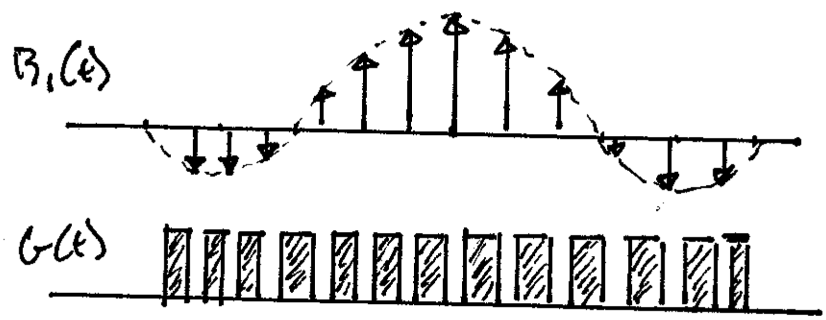
$$\underline{n}_i = \frac{\gamma \Delta t}{|\theta_i|} \left(B_{1,x}(t_i),\ B_{1,y}(t_i),\ Gx \right)$$

WE CAN THEN COMPUTE (α_i, β_i) AND Q_i, AND

$$Q = Q_n Q_{n-1} \ldots Q_2 Q_1$$

HARD PULSE APPROXIMATION

TREMENDOUS SIMPLIFICATION IF WE ASSUME RF CONSISTS OF IMPULSES SEPARATED BY FREE PRECESSION INTERVALS



GOOD APPROXIMATION TO CONTINUOUS "SOFT" PULSE IF ROTATIONS ARE SMALL

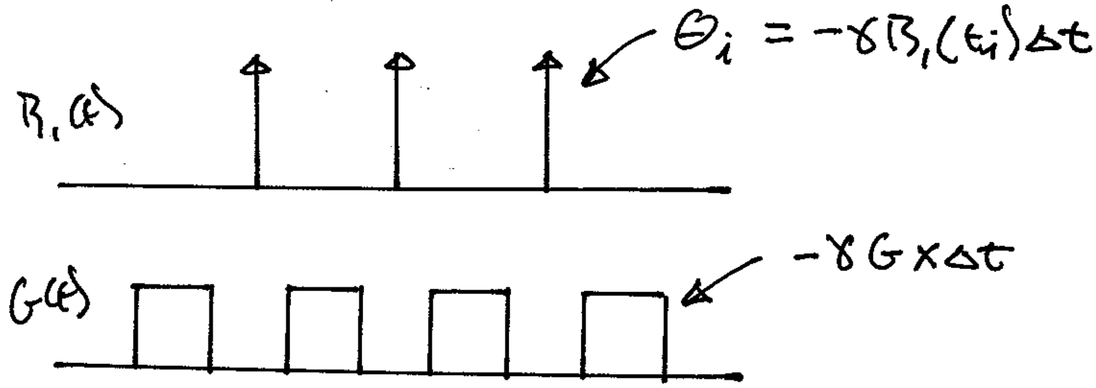
ARBITRARILY GOOD IF YOU SAMPLE FINELY ENOUGH

DUE TO

MEIR SHINNAR (+ JACK LEIGH)

PATRICK LE ROUX

ZOOMED IN VIEW



THE INCREMENTAL ROTATION MATRIX IS

$$Q_i = \underbrace{\begin{pmatrix} C_i & -S_i^* \\ S_i & C_i \end{pmatrix}}_{\text{HARD PULSE ROTATION}} \underbrace{\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}}_{\text{FREE PRECESSION}}$$

WHERE

$$C_i = \cos(\gamma |B_1(t_i)| \Delta t/2)$$

$$S_i = i \underbrace{e^{i \angle B_1(t_i)}}_{(n_x + i n_y)} \overbrace{\sin(\gamma |B_1(t_i)| \Delta t/2)}^{n_z = 0}$$

$$z = e^{i \gamma G x \Delta t}$$

IF WE SUBSTITUTE INTO THE RECURSION

$$\begin{pmatrix}\alpha_i\\ \beta_i\end{pmatrix} = \begin{pmatrix}C_i & -S_i^*\\ S_i & C_i\end{pmatrix}\begin{pmatrix}z^{1/2} & 0\\ 0 & z^{-1/2}\end{pmatrix}\begin{pmatrix}\alpha_{i-1}\\ \beta_{i-1}\end{pmatrix}$$

$$= z^{1/2}\begin{pmatrix}C_i & -S_i^*\\ S_i & C_i\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & z^{-1}\end{pmatrix}\begin{pmatrix}\alpha_{i-1}\\ \beta_{i-1}\end{pmatrix}$$

WE WANT TO GET RID OF HALF POWERS OF z, SO DEFINE

$$A_i = z^{-i/2}\alpha_i$$

$$B_i = z^{-i/2}\beta_i$$

THEN

$$\begin{pmatrix}A_i\\ B_i\end{pmatrix} = \begin{pmatrix}C_i & -S_i^*\\ S_i & C_i\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & z^{-1}\end{pmatrix}\begin{pmatrix}A_{i-1}\\ B_{i-1}\end{pmatrix}$$

THE INITIAL CONDITION IS NO ROTATION, SO

$$\begin{pmatrix}A_0\\ B_0\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$$

THEN

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} C_1 & -S_1^* \\ S_1 & C_1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$= \begin{pmatrix} C_1 \\ S_1 \end{pmatrix}$$

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} C_2 & -S_2^* \\ S_2 & C_2 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} C_1 \\ S_1 \end{pmatrix}}_{\begin{pmatrix} C_1 \\ S_1 z^{-1} \end{pmatrix}}$$

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} C_1 C_2 - S_1 S_2^* z^{-1} \\ C_1 S_2 + S_1 C_2 z^{-1} \end{pmatrix}$$

POLYNOMIALS IN z^{-1}!

AT THE n^{th} TIME STEP

$$A_N(z) = \sum_{j=0}^{n-1} A_{N,j} z^{-j}$$

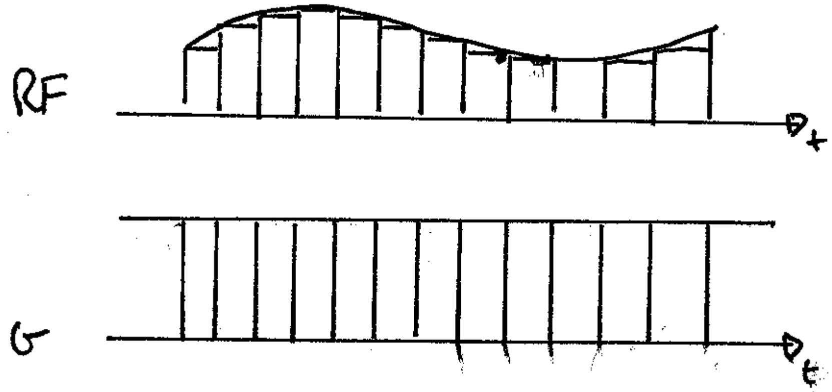
$$B_N(z) = \sum_{j=0}^{n-1} B_{N,j} z^{-j}$$

TWO $(n-1)$ ORDER POLYNOMIALS IN $z = e^{i \delta \gamma x \sigma t}$.

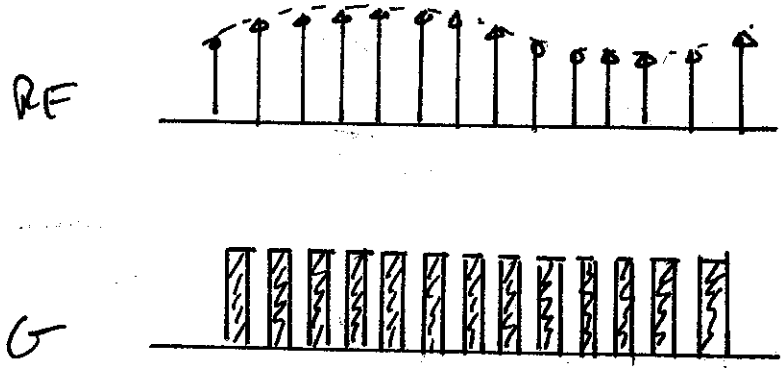
<u>FORWARD SLR TRANSFORM</u>

FORWARD SLR TRANSFORM

APPROXIMATE A "SOFT" RF PULSE



BY ALTERNATING SEQUENCE OF "HARD" PULSES AND FREE PRECESSION GRADIENT INTERVALS



ONE INCREMENTAL ROTATION IS

$$Q_i = \underbrace{\begin{pmatrix} C_i & -S_i^* \\ S_i & C_i \end{pmatrix}}_{\text{HARD PULSE}} \underbrace{\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}}_{\text{FREE PRECESSION}}$$

$$C_i = \cos(\gamma |B_1(t_i)| \Delta t/2)$$

$$S_i = i e^{i\angle B_1(t_i)} \sin(\gamma |B_1(t_i)| \Delta t/2)$$

$$z = e^{-i\gamma G x \Delta t}$$

RECURSION FOR α, β

$$\begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} = \begin{pmatrix} c_j & -s_j^* \\ s_j & c_j \end{pmatrix} \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix} \begin{pmatrix} \alpha_{j-1} \\ \beta_{j-1} \end{pmatrix}$$

$$= z^{1/2} \begin{pmatrix} c_j & -s_j^* \\ s_j & c_j \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} \alpha_{j-1} \\ \beta_{j-1} \end{pmatrix}$$

DEFINE

$$A_j = z^{-j/2} \alpha_j$$

$$B_j = z^{-j/2} \beta_j$$

THEN

$$\begin{pmatrix} A_j \\ B_j \end{pmatrix} = \begin{pmatrix} c_j & -s_j^* \\ s_j & c_j \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} A_{j-1} \\ B_{j-1} \end{pmatrix}$$

STARTING WITH $(A_0, B_0)^T = (1, 0)^T$, WE GET

$$\begin{pmatrix} A_N(z) \\ B_N(z) \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{N-1} A_{N,j} z^{-j} \\ \sum_{j=0}^{N-1} B_{N,j} z^{-j} \end{pmatrix}$$

TWO $(N-1)$ ORDER POLYNOMIALS IN

$$z = e^{-i\gamma G x \Delta t}$$

INVERSE SLR TRANSFORM

REMARKABLE FACT

GIVEN $A_N(z)$ AND $B_N(z)$, THE SLR TRANSFORM CAN BE INVERTED TO PRODUCE $B_1(t)$

$\Rightarrow$ IF WE CAN DESIGN $A_N(z)$ AND $B_N(z)$ WE CAN DESIGN $B_1(t)$.

MAGNITUDE CONSTRAINT

$$|A_N(z)|^2 + |B_N(z)|^2 = 1 \qquad z = e^{-i\gamma G x \delta t}$$

$(A_N(z), B_N(z))^T$ MUST BE A VALID ROTATION FOR ANY $|z| = 1$

BACK RECURSION

ONE STEP OF THE FORWARD SLR TRANSFORM

$$\underbrace{\begin{pmatrix} A_j \\ B_j \end{pmatrix}}_{\substack{j-1 \text{ ORDER} \\ \text{POLYNOMIALS}}} = \underbrace{\begin{pmatrix} C_j & -S_j^* z^{-1} \\ S_j & C_j z^{-1} \end{pmatrix}}_{\substack{\text{UNITARY} \\ \text{MATRIX} \\ Q_j}} \underbrace{\begin{pmatrix} A_{j-1} \\ B_{j-1} \end{pmatrix}}_{\substack{j-2 \text{ ORDER} \\ \text{POLYNOMIALS}}}$$

INVERSE RECURSION

$$\begin{pmatrix} A_{j-1} \\ B_{j-1} \end{pmatrix} = \underbrace{\begin{pmatrix} c_j & s_j^* \\ -s_j z & c_j z \end{pmatrix}}_{Q_j^* = Q_j^{-1}} \begin{pmatrix} A_j' \\ B_j' \end{pmatrix}$$

$$\begin{pmatrix} A_{j-1} \\ B_{j-1} \end{pmatrix} = \begin{pmatrix} c_j A_j' + s_j^* B_j' \\ z\left(-s_j A_j' + c_j B_j'\right) \end{pmatrix}$$

WE KNOW $(A_j', B_j')^T$ AT EACH STAGE OF THE BACK RECURSION

ALSO, WE KNOW $(A_{j-1}, B_{j-1})^T$ ARE LOWER ORDER THAN $(A_j, B_j)^T$

$\Rightarrow$ LEADING TERM OF A_{j-1} MUST DROP OUT

$\Rightarrow$ TRAILING TERM OF B_{j-1} MUST DROP OUT

$$C_j A_{j,j-1} + S_j^* B_{j,j-1} = 0$$ LEADING COEFFICIENTS

$$-S_j A_{j,0} + C_j B_{j,0} = 0$$ TRAILING COEFFICIENTS

APPEAR TO BE TWO INDEPENDENT CONDITIONS, BUT ARE IN FACT THE SAME. FROM THE MAGNITUDE CONSTRAINT

$$|A_N(z)|^2 + |B_N(z)|^2 = 1$$

WE CAN SHOW THAT

$$A_{j,j-1} A_{j,0}^* + B_{j,j-1} B_{j,0}^* = 0$$

WITH THIS, EITHER OF THE CONSTRAINTS CAN BE DERIVED FROM THE OTHER.

CHOOSING THE LOW ORDER RECURSION

$$-S_j A_{j,0} + C_j B_{j,0} = 0$$

$$S_j A_{j,0} = C_j B_{j,0}$$

$$\frac{B_{j,0}}{A_{j,0}} = \frac{S_j}{C_j}$$

$$= \frac{i e^{i\phi_j} \sin \theta_j/2}{\cos \theta_j/2}$$

$$= i e^{i\phi_j} \tan \theta_j/2$$

THEN

$$\theta_j = 2 \tan^{-1}\left(\left|\frac{B_{j,0}}{A_{j,0}}\right|\right)$$

$$\phi_j = \angle\left(-i\, B_{j,0}/A_{j,0}\right)$$

THE RF WAVEFORM IS

$$\underline{B_1(t_j) = \frac{1}{\gamma \Delta t} \theta_j e^{i\phi_j}}$$

<u>INVERSE SLR TRANSFORM</u>

SLR TRANSFORM

INVERTABLE RELATION BETWEEN

$$B_1(t) \overset{SLR}{\Longleftrightarrow} (A_N(z), B_N(z))$$

SAME STRUCTURE TURNS UP IN MANY OTHER PLACES

"LAYER PEELING" ALGORITHMS

WAVE PROPOGATION THROUGH INHOMOGENEOUS MEDIA (SEISMOLOGY)

QMF FILTERS

QUADRATURE MIRROR FILTERS

PERFECT MULTIBAND DECIMATION AND RECONSTRUCTION FILTERS

LATTICE FILTERS

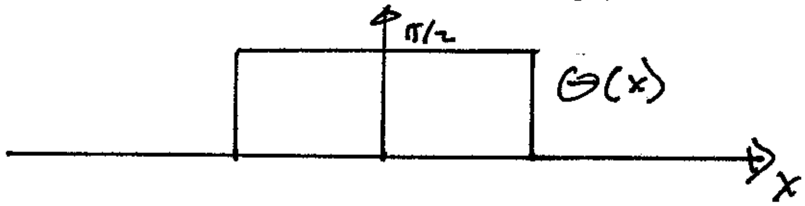
SPECIAL CASE OF A LATTICE FILTER WHERE EACH STAGE IS A EUCLIDEAN ROTATION.

GOOD QUANTIZATION AND DYNAMIC RANGE PROPERTIES

RF PULSE DESIGN WITH SLR

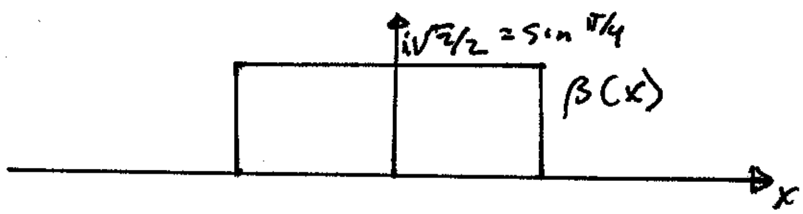
BASIC ALGORITHM ($(\pi/2)_x$ PULSE EXAMPLE)

1) CHOOSE A FLIP ANGLE PROFILE AS A FUNCTION OF SPACE



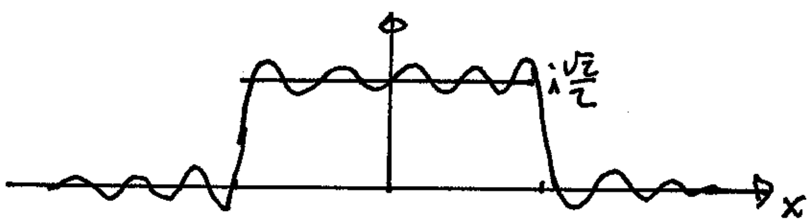
IDEAL FLIP-ANGLE PROFILE EXCITATION EXAMPLE

2) COMPUTE IDEAL $\beta(x) = \sin \Theta(x)/2$



IDEAL β PROFILE

3) APPROXIMATE IDEAL β WITH $B_N(e^{i\gamma G x x \Delta t})$



$B_N(z)\Big|_{z=e^{i\gamma G x x \Delta t}}$

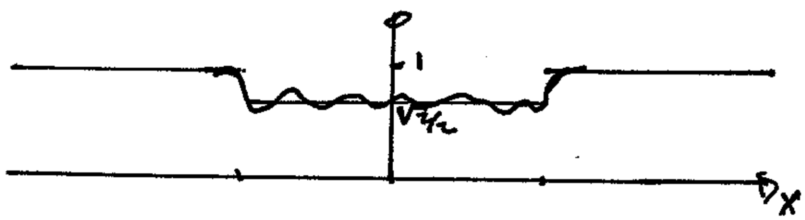
LOWPASS DISCRETE-TIME FILTER

SAMPLED SMALL-TIP-ANGLE FOURIER DESIGN RF PULSE (WINDOWED SINC)

4) USE THE MAGNITUDE CONSTRAINT

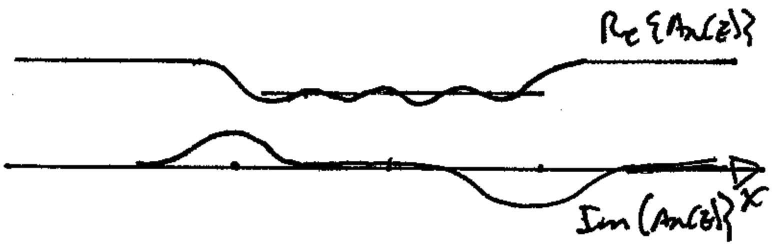
$$|A_N(z)|^2 + |B_N(z)|^2 = 1 \qquad z = e^{-i\gamma G x \Delta t}$$

TO SOLVE FOR $|A_N(z)| = \sqrt{1 - |B_N(z)|^2}$



α PROFILE

5) SOLVE FOR THE PHASE OF $A_N(z)$, AND HENCE $A_N(z)$



$A_N(z)$ NOT UNIQUE

MOST USEFUL SOLUTION:

<u>MINIMUM PHASE $A_N(z)$</u>

EASY TO COMPUTE

MINIMUM INTEGRATED POWER

6) USE INVERSE SLR RECURSION TO PRODUCE RF PULSE

MINIMUM PHASE $A_N(Z)$

WRITE

$$A_N(Z) = |A_N(Z)|\, e^{i \angle A_N(Z)}$$

COMPLEX LOGARITHM IS

$$\log A_N(Z) = \log|A_N(Z)| + i \angle A_N(Z)$$

IF $A_N(Z)$ IS MINIMUM PHASE, NO ZEROS OR POLES ON OR OUTSIDE UNIT CIRCLE, THEN

$$\log A_N(Z)$$

IS AN ANALYTIC SIGNAL (ZERO FOR NEGATIVE TIME, THE OTHER DOMAIN)

IN THIS CASE

$$\angle A_N(Z) = \mathcal{H}\{\log|A_N(Z)|\}$$

WHICH WE CAN COMPUTE DIRECTLY. THEN

$$A_N(Z) = |A_N(Z)|\, e^{i \mathcal{H}\{\log|A_N(Z)|\}}$$

MINIMUM RF POWER

MINIMUM PHASE $A_N(z)$ HAS THE LARGEST CONSTANT COEFFICIENT $A_{N,0}$.

THE FORWARD RECURSION IS

$$\begin{pmatrix} A_j(z) \\ B_j(z) \end{pmatrix} = \begin{pmatrix} C_j & -S_j^* z^{-1} \\ S_j & C_j z^{-1} \end{pmatrix} \begin{pmatrix} A_{j-1}(z) \\ B_{j-1}(z) \end{pmatrix}$$

THE CONSTANT COEFFICIENT IS THEN

$$A_{N,0} = C_N C_{N-1} \cdots C_2 C_1$$

FOR SMALL INCREMENTAL TIP ANGLES

$$C_j = \cos \theta_j/2 \simeq 1 - \tfrac{1}{2}\left(\tfrac{\theta_j}{2}\right)^2$$

$$= 1 - \tfrac{1}{8}\theta_j^2$$

THEN

$$A_{N,0} = \left(1 - \tfrac{1}{8}\theta_n^2\right)\left(1 - \tfrac{1}{8}\theta_{n-1}^2\right) \cdots \left(1 - \tfrac{1}{8}\theta_2^2\right)\left(1 - \tfrac{1}{8}\theta_1^2\right)$$

$$= 1 - \underbrace{\tfrac{1}{8}\sum_{j=0}^{N}\theta_n^2}_{\sim \frac{1}{N}} + \underbrace{\cdots}_{\text{HIGHER ORDER TERMS } \sim \frac{1}{N^2} \text{ AND FASTER}}$$

RECALL

$$\theta_j = \gamma |B_{1,j}| \Delta t$$

SO

$$A_{N,0} = 1 - \frac{1}{8} (\gamma \Delta t)^2 \underbrace{\sum_{j=0}^{N} |B_{1,j}|^2}_{\text{RF POWER}}$$

LARGEST $A_{N,0}$ MEANS SMALLEST RF POWER

MINIMUM PHASE $A_N(z)$ GIVES MINIMUM POWER $B_1(t)$

ALMOST ALWAYS WHAT YOU WANT

PULSE DESIGN IS DETERMINED BY $B_N(z)$

$A_N(z)$ IS CHOSEN TO BE CONSISTANT, MINIMUM POWER

EXCEPTION SELF REFOCUSING PULSES

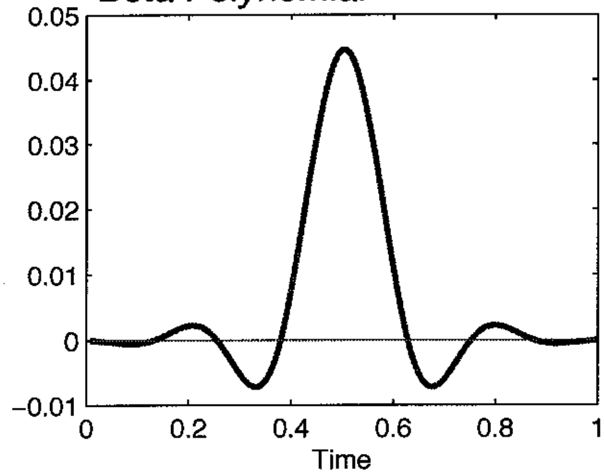
PHASE ADDED TO α SO THAT

$$2\alpha^* \beta$$

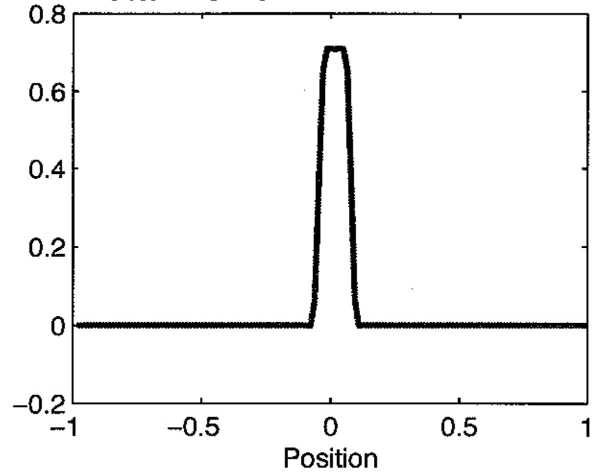
HAS ENOUGH PHASE TO SHIFT THE ECHO TO THE END OF THE PULSE OR BEYOND. VERY EXPENSIVE IN RF POWER!

SLR Excitation Pulse Design

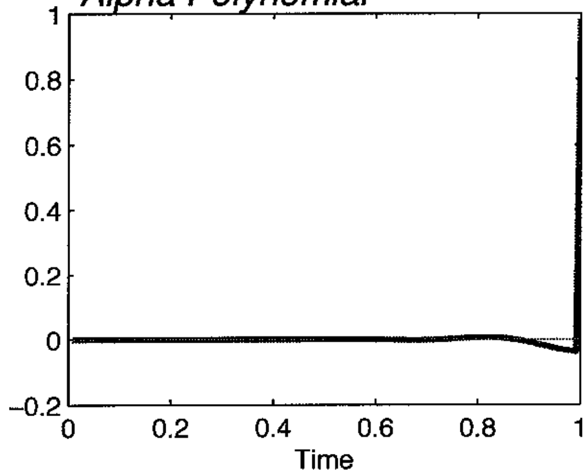
Beta Polynomial



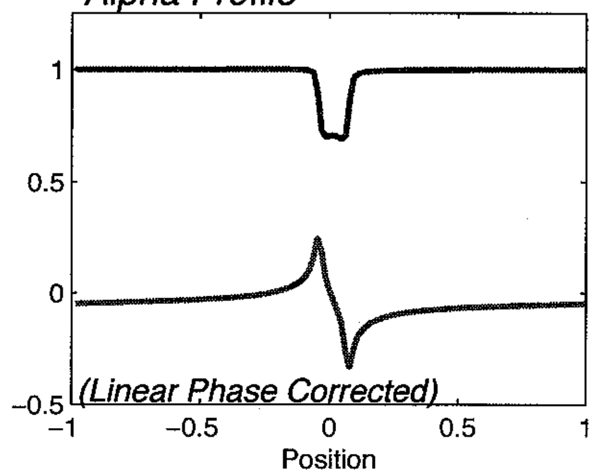
Beta Profile



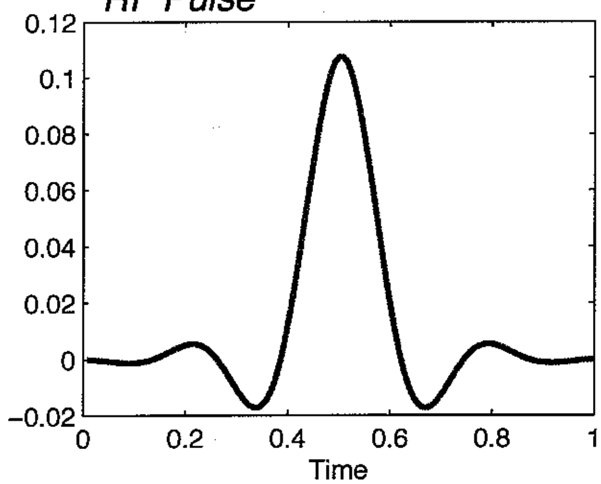
Alpha Polynomial



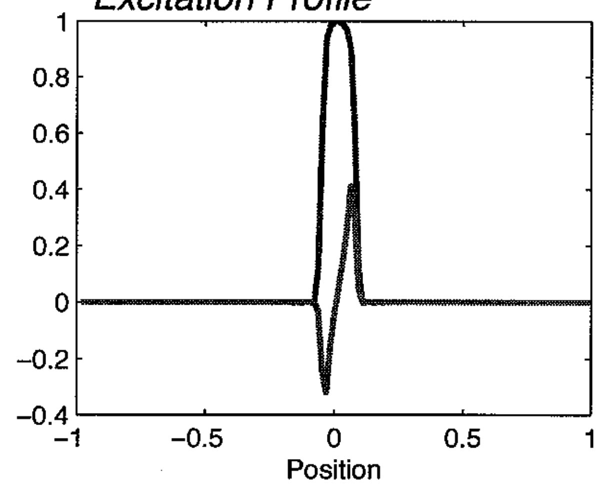
Alpha Profile



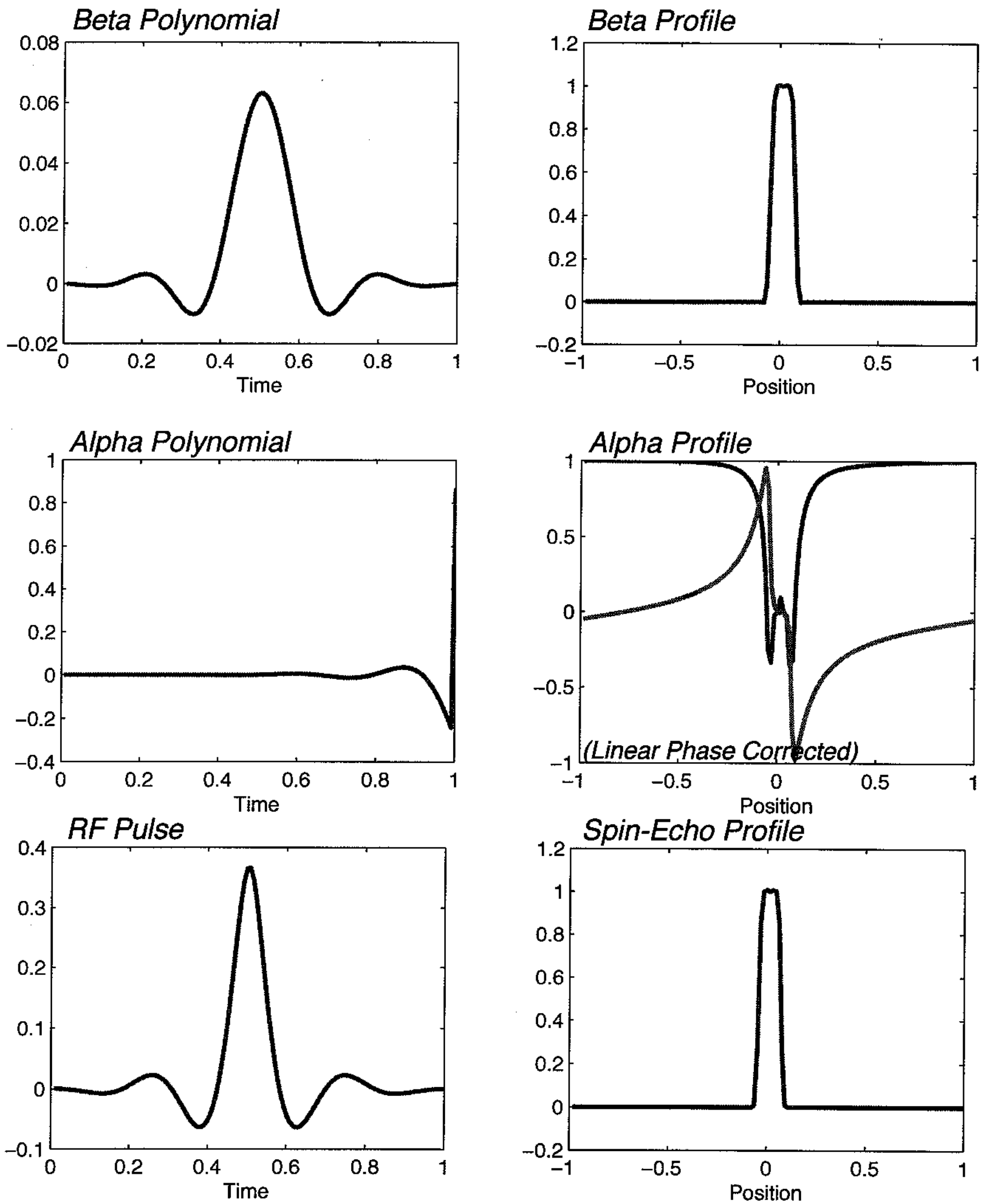
RF Pulse



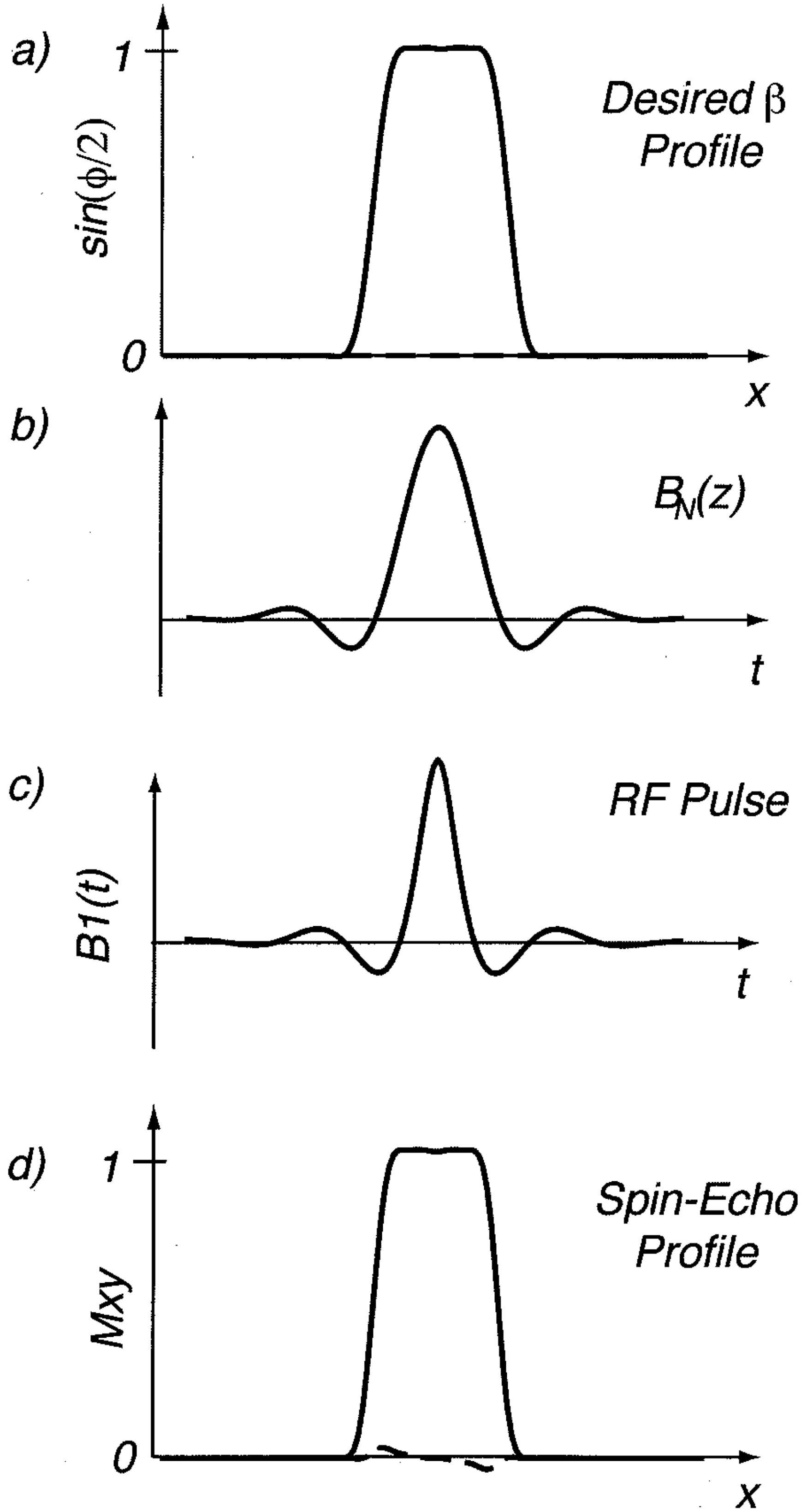
Excitation Profile



SLR Spin-Echo Pulse Design



SLR Spin-Echo Pulse Design



TYPES OF $B_N(z)$ DESIGNS

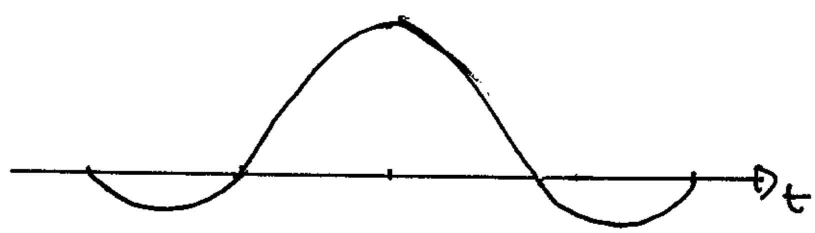
MANY DIFFERENT OPTIONS FOR $B_N(z)$

LINEAR PHASE: MOST COMMON

PERFECTLY REFOCUSED WITH GRADIENT REVERSAL AS AN EXCITATION PULSE

SPIN ECHO PULSES

SYMMETRIC IN TIME

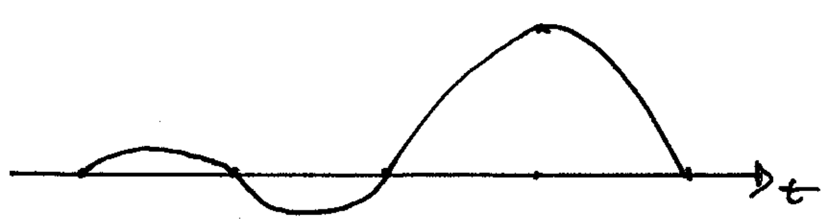


ALSO MAXIMUM PEAK POWER (PERFECTLY REPHASES!)

NOT THE MOST SELECTIVE

MINIMUM PHASE SAT PULSES AND INVERSIONS

THE FLIP OCCURS AS LATE IN THE PULSE AS POSSIBLE



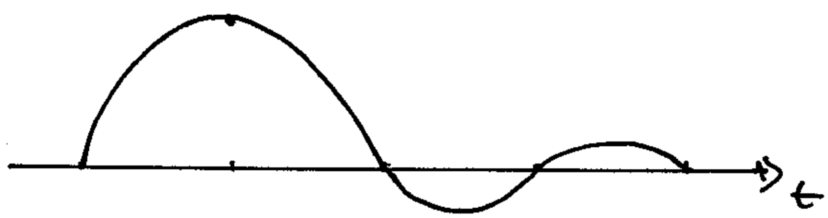
MOST SELECTIVE PULSES

DOES NOT PERFECTLY REFOCUS

ALMOST THE SAME PEAK POWER AS LINEAR PHASE

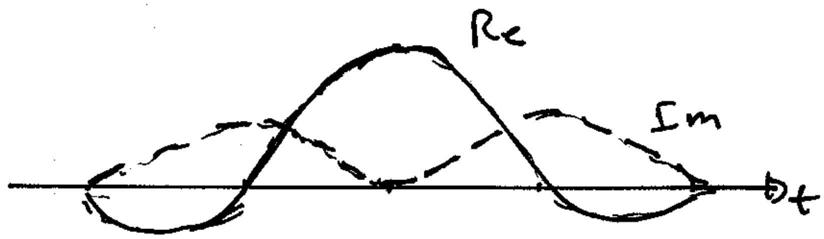
<u>MAXIMUM PHASE</u> SATURATION AND INVERSION

MINIMUM PHASE PULSE REVERSED



<u>QUADRATIC OR NONLINEAR PHASE</u>

SPREADS RF POWER OUT



IDENTICAL TOTAL POWER AS MIN/MAX PHASE PULSE WITH SAME PROFILE

<u>MUCH</u> LOWER PEAK POWER

BASIC PROBLEM

RF PULSE DESIGN WITH SLR CONSISTS OF:

1) DESIGNING $B_N(z)$

2) CHOOSING A COMPATIBLE, MINIMUM POWER $A_N(z)$.

3) PERFORMING THE BACK RECURSION

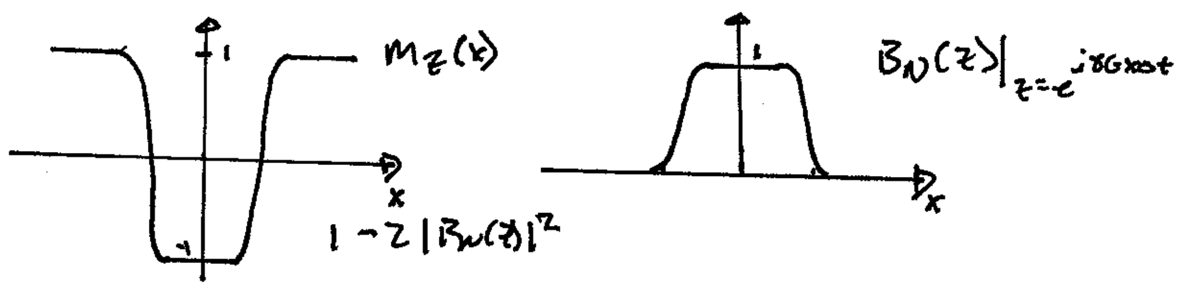
ALL DETERMINED ONCE $B_N(z)$ HAS BEEN DESIGNED.

HOW DO WE DESIGN $B_N(z)$?

SEVERAL ISSUES:

1) GOAL IS SPECIFIED IN TERMS OF M_{xy}, OR M_z

$B_N(z)$ IS $\sim \sin(\theta(x)/2)$



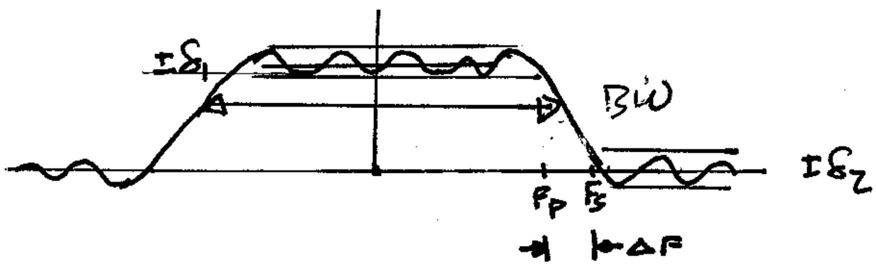
THE GOAL IS A NON-LINEAR FUNCTION OF THE INPUT $B_N(z)$.

WE NEED TO FIGURE OUT WHAT TO ASK FOR TO GET WHAT WE WANT.

2) THE PARAMETERS WE CARE ABOUT ARE

PASSBAND ERROR (δ_1)
STOPBAND ERROR (δ_2)
SLICE WIDTH IN FREQUENCY (BW)
PULSE LENGTH (T)

GIVEN THESE WE WOULD LIKE MINIMUM TRANSITION WIDTH $(F_S - F_P) = \Delta F$



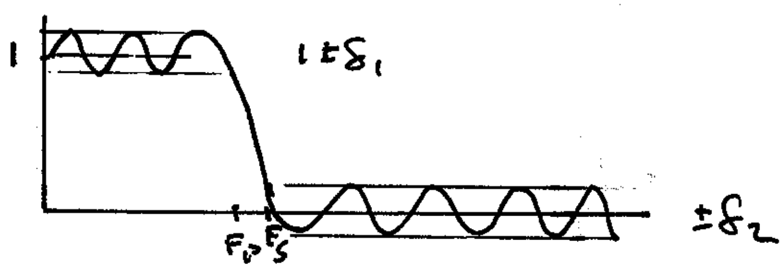
WHAT FILTER DESIGN PROGRAMS WANT IS

PASSBAND EDGE (F_P)
STOPBAND EDGE (F_S)
PASSBAND/STOPBAND ERROR RATIO (δ_1/δ_2)

WE NEED TO RELATE WHAT WE WANT TO WHAT WE NEED TO SPECIFY.

BASIC IDEA

EQUAL-RIPPLE FILTERS (PARKS-McCLELLAN) ARE DETERMINED BY BAND EDGES AND RIPPLE AMPLITUDES



ALL WE HAVE TO DO IS FIGURE OUT WHAT THE EFFECTIVE RIPPLE PRODUCED IN THE SLICE PROFILE OF INTEREST IS.

δ_1^e – PASSBAND MAGNETIZATION RIPPLE

δ_2^e – STOPBAND MAGNETIZATION RIPPLE

THIS WILL DEPEND ON THE PROFILE. ONCE WE HAVE THESE RELATIONS, WE CAN INVERT THEM TO DETERMINE WHAT (δ_1, δ_2) TO SPECIFY.

EXAMPLE: INVERSION PULSES

INVERSION PROFILE

$$M_z^+(x) = \left(1 - 2|\beta(x)|^2\right) M_0$$
$$= \left(1 - 2|B_N(z)|^2\right) M_0 \Big|_{z = e^{i\gamma G x \Delta t}}$$

THIS CASE WE CAN ACTUALLY SOLVE FOR EXPLICITLY BY DESIGNING $M_z^+(x)$, AND FACTORING IT. WE WILL RETURN TO THIS.

OUT-OF-SLICE RIPPLE

AN INPUT RIPPLE OF δ_2 PRODUCES

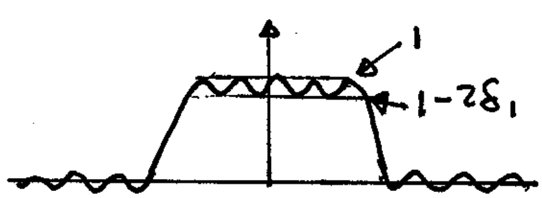
$$\delta_2^e = 2\delta_2^2$$

SO

$$\underline{\delta_2 = \sqrt{\delta_2^e / 2}}$$

IN-SLICE RIPPLE

$B_N(z)$ IS SCALED TO BE LESS THAN 1



MAXIMUM RIPPLE OCCURS IN M_z WHEN $B_N(z)$ IS MINIMUM

$$M_z = \left(1 - 2(1-2\delta_1)^2\right) M_0$$
$$= \left(1 - 2(1 - 4\delta_1 + 4\delta_1^2)\right) M_0$$
$$\simeq (-1 + 8\delta_1) M_0$$

SO

$$\delta_1^e = 8\delta_1$$
$$\delta_1 = \tfrac{1}{8}\delta_1^e$$

FOR EXAMPLE, IF WE WANT AN INVERSION PROFILE WITH $\delta_1^e = 0.01$ AND $\delta_2^e = 0.01$, WE NEED TO DESIGN $B_N(z)$ WITH

$$\delta_2 = \sqrt{\delta_2^e/2} = \sqrt{0.01/2} = 0.07 \quad \text{MUCH LARGER!}$$
$$\delta_1 = \tfrac{1}{8}\delta_1^e = \frac{0.01}{8} = 0.0013 \quad \text{MUCH SMALLER!}$$

MORE IMPORTANTLY, THE RATIO

$$\frac{\delta_2}{\delta_1} = 53$$

FAR FROM THE UNITY RATIO OF M_z.

SIMILAR RELATIONS CAN BE DERIVED FOR OTHER TYPES OF PULSES

CASE	δ_1	δ_2	
SMALL TIP	δ_1^e	δ_2^e	
$\pi/2$	$\sqrt{\delta_1^e/2}$	$\delta_2^e/\sqrt{2}$	(NOT USABLE)
INVERSION	$\delta_1^e/8$	$\sqrt{\delta_2^e/2}$	
SPIN ECHO	$\delta_1^e/4$	$\sqrt{\delta_2^e}$	
SATURATION	$\delta_1^e/2$	$\sqrt{\delta_2^e}$	

FROM PAULY, LE ROUX, et al, IEEE TMI 10(1), p 53-65, 1991

WE KNOW RIPPLE AMPLITUDES δ_1, δ_2

HOW DO WE FIND PASSBAND EDGES?

FROM DIGITAL FILTER DESIGN

$$\underbrace{T}_{\text{DURATION}} \underbrace{(\Delta F)}_{\substack{\text{TRANSITION} \\ \text{IN HZ}}} = \underbrace{D_\infty(\delta_1, \delta_2)}_{\substack{\text{CONSTANT, FCN OF} \\ \delta_1 \ \delta_2}}$$

OR

$$\underbrace{T(BW)}_{\substack{\text{TIME} \\ \text{BANDWIDTH}}} \left(\underbrace{\frac{\Delta F}{BW}}_{\substack{\text{FRACTIONAL} \\ \text{TRANSITION} \\ \text{WIDTH}}}\right) = D_\infty(\delta_1, \delta_2)$$

D_∞ HAS BEEN DETERMINED EMPIRICALLY FOR EQUI-RIPPLE FILTERS

INTUITIVELY WE EXPECT

$$\Delta F \quad \frac{1}{T}$$

FOR A SINC, AND

$$\Delta F \sim \frac{2}{T}$$

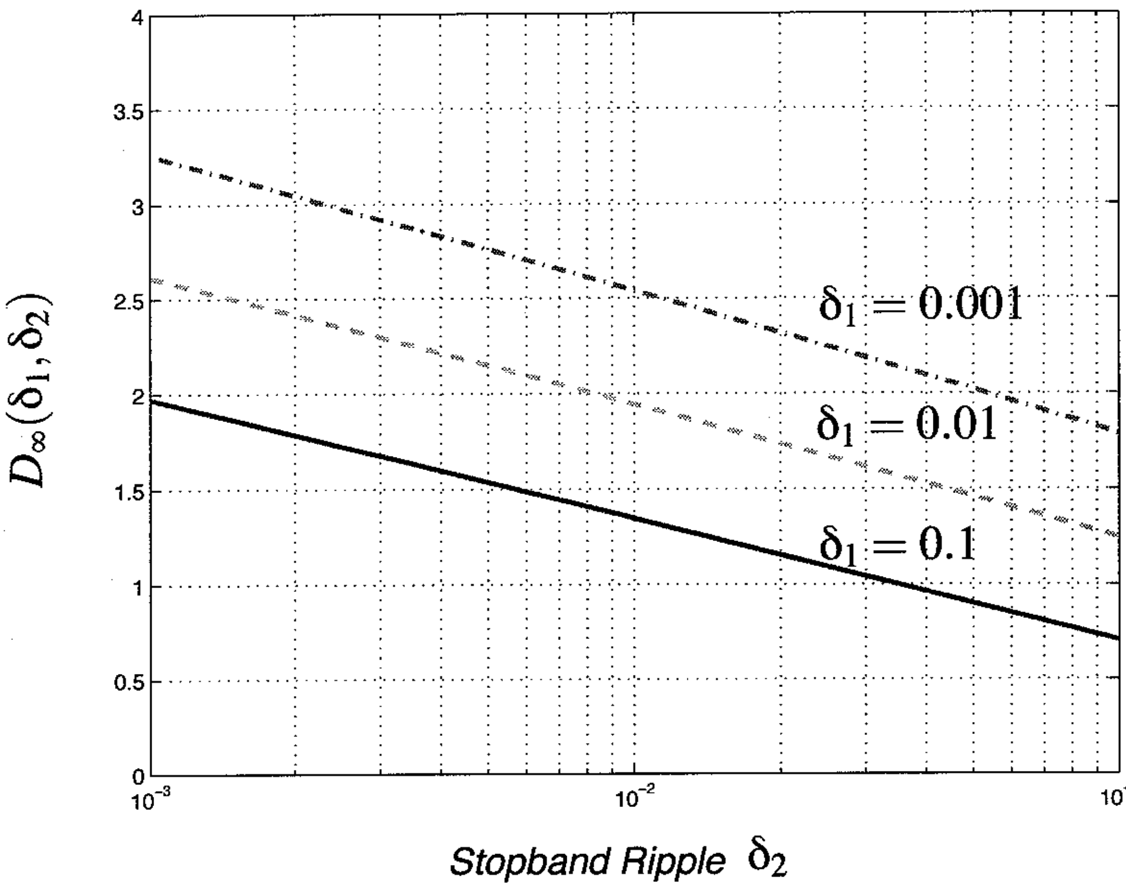
FOR A WINDOWED SINC

SO D_∞ SHOULD BE ON THE ORDER OF 1 TO 2, THIS IS A GOOD ESTIMATE!

BETTER FILTER DESIGNS HAVE LOWER D_∞

LIMIT TO HOW MUCH CAN BE GAINED.

$$D_\infty(\delta_1, \delta_2)$$



$$D_\infty(\delta_1, \delta_2) = (a_1 L_1^2 + a_2 L_1 + a_3)L_2 + (a_4 L_1^2 + a_5 L_1 + a_6)$$

Where

$$L_1 = \log_{10}\delta_1 \quad \text{and} \quad L_2 = \log_{10}\delta_2$$

and

$$a_1 = 5.309 \times 10^{-3}$$
$$a_2 = 7.114 \times 10^{-2}$$
$$a_3 = -4.761 \times 10^{-1}$$
$$a_4 = -2.66 \times 10^{-3}$$
$$a_5 = -5.941 \times 10^{-1}$$
$$a_6 = -4.278 \times 10^{-1}$$

DESIGN EXAMPLE

INVERSION PULSE

$T = 4\,ms$

$BW = 2\,KHZ$ $\qquad\qquad T(BW) = 8$

$\delta_1^e = 0.01$

$\delta_2^e = 0.01$

FROM PREVIOUS EXAMPLE

$$\delta_1 = \delta_1^e / 8 = 0.00125$$

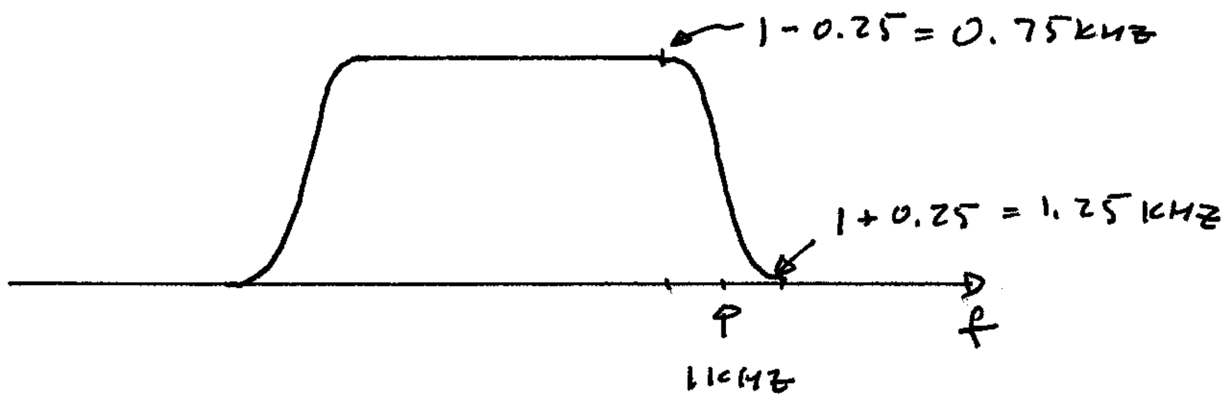
$$\delta_2 = \sqrt{\delta_1^e / 2} = 0.0707$$

THEN

$$D_\infty(0.00125, 0.0707) = 2$$

THE TRANSITION WIDTH IS THEN

$$\Delta F = \frac{D_\infty(\delta_1, \delta_2)}{T} = \frac{2}{4\,ms} = 500\,HZ$$



IN MATLAB, DESIGN THIS FILTER WITH

firpm.m

(WAS remez.m)

INPUTS

(SAMPLING FREQ)/2
256 SAMPLES IN 4ms

$$f = [0 \quad 750 \quad 1250 \quad 32000]/32000;$$

$$m = [1 \quad 1 \quad 0 \quad 0]$$

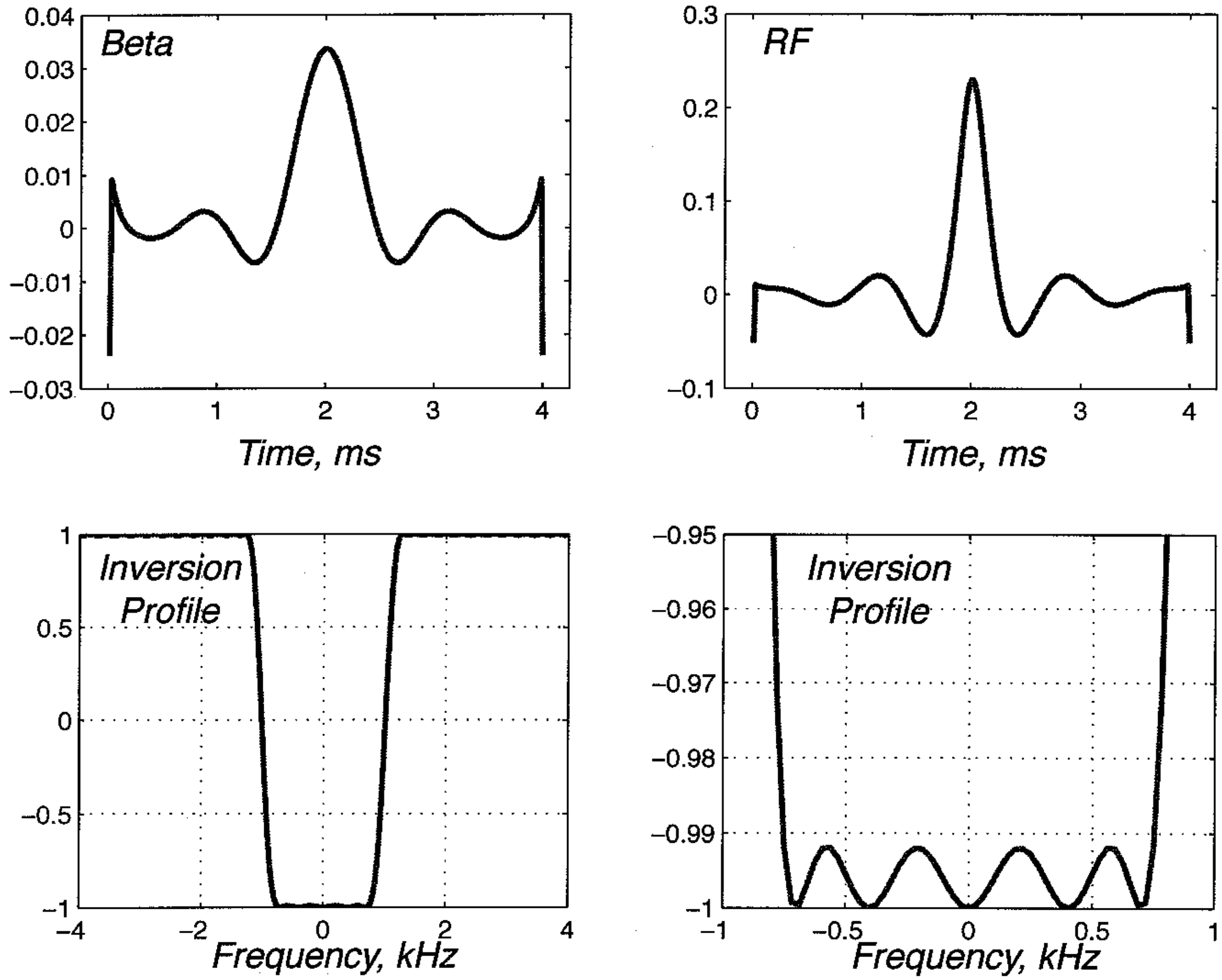
$$w = [1 \quad \delta_1/\delta_2] = [1 \quad 0.0177]$$

THEN

$$b = \text{firpm}(\overset{N-1}{255}, f, m, w)$$

THIS IS BETA. SCALE TO $\sin(\theta/2)$, THEN APPLY INVERSE SLR.

PM Inversion



DRAWBACKS OF PM DESIGNS

1) LARGE SPIKES COMMON AT FIRST/LAST SAMPLES (CONOLLY WINGS)

2) SPIKES GET LARGER AS N INCREASES

3) INTEGRATED ABSOLUTE VALUE OF STOPBAND ($|1-|m_2|$ FOR EXAMPLE) CAN BE LARGE.

ANOTHER ALTERNATIVE IS WEIGHTED LEAST SQUARES

$$b = \text{firls}(N, f, m, w)$$

SAME INPUTS AS REMEZ.

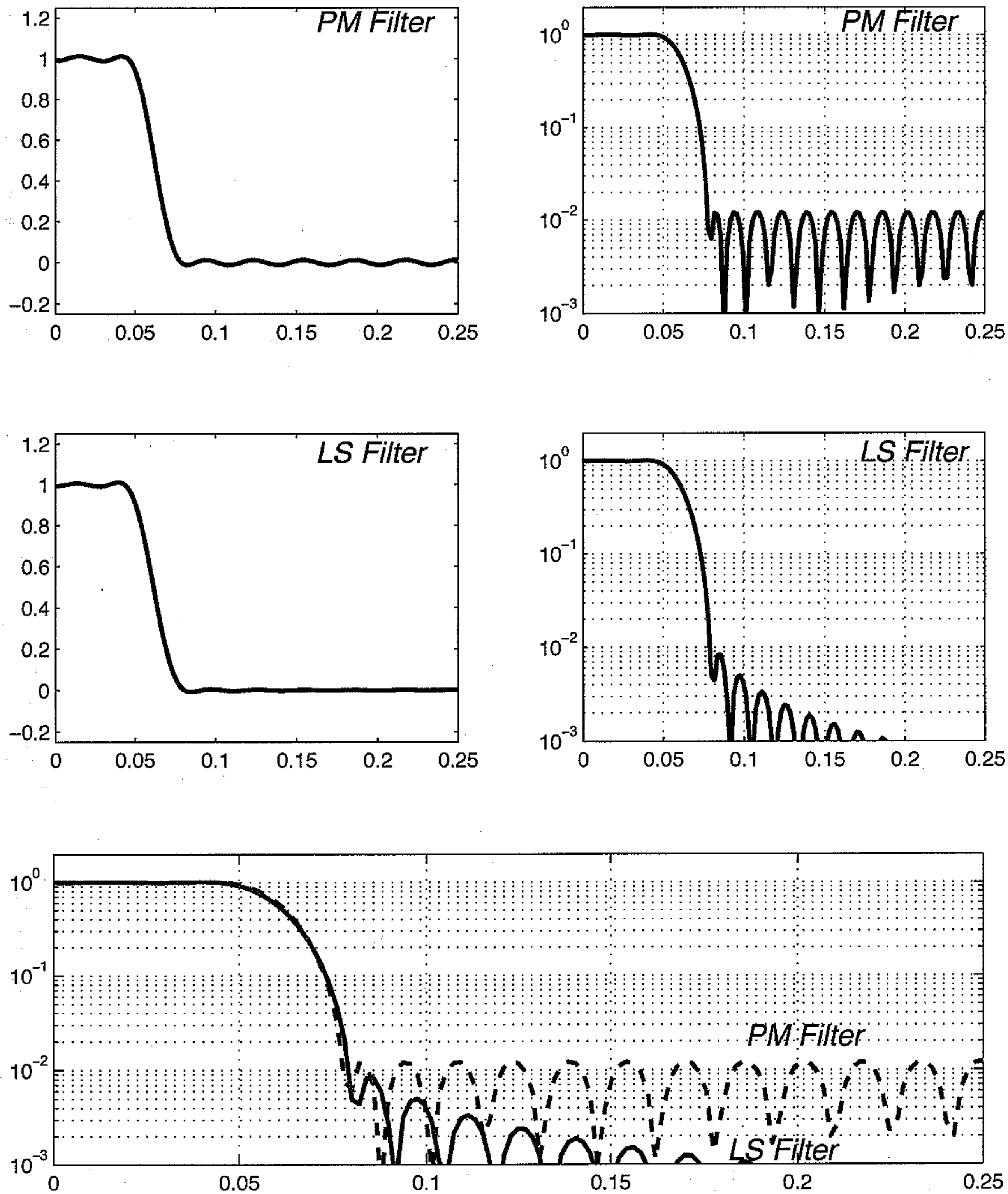
$D_\infty(\delta_1, \delta_2)$ WILL BE DIFFERENT FOR FIRLS, BUT NOT KNOWN.

FORTUNATELY THE $D_\infty(\delta_1, \delta_2)$ FOR PM FILTERS IS REASONABLY CLOSE.

<u>FIRLS DESIGNS ARE RECOMMENDED</u> UNLESS THERE ARE OTHER IMPORTANT FACTORS.

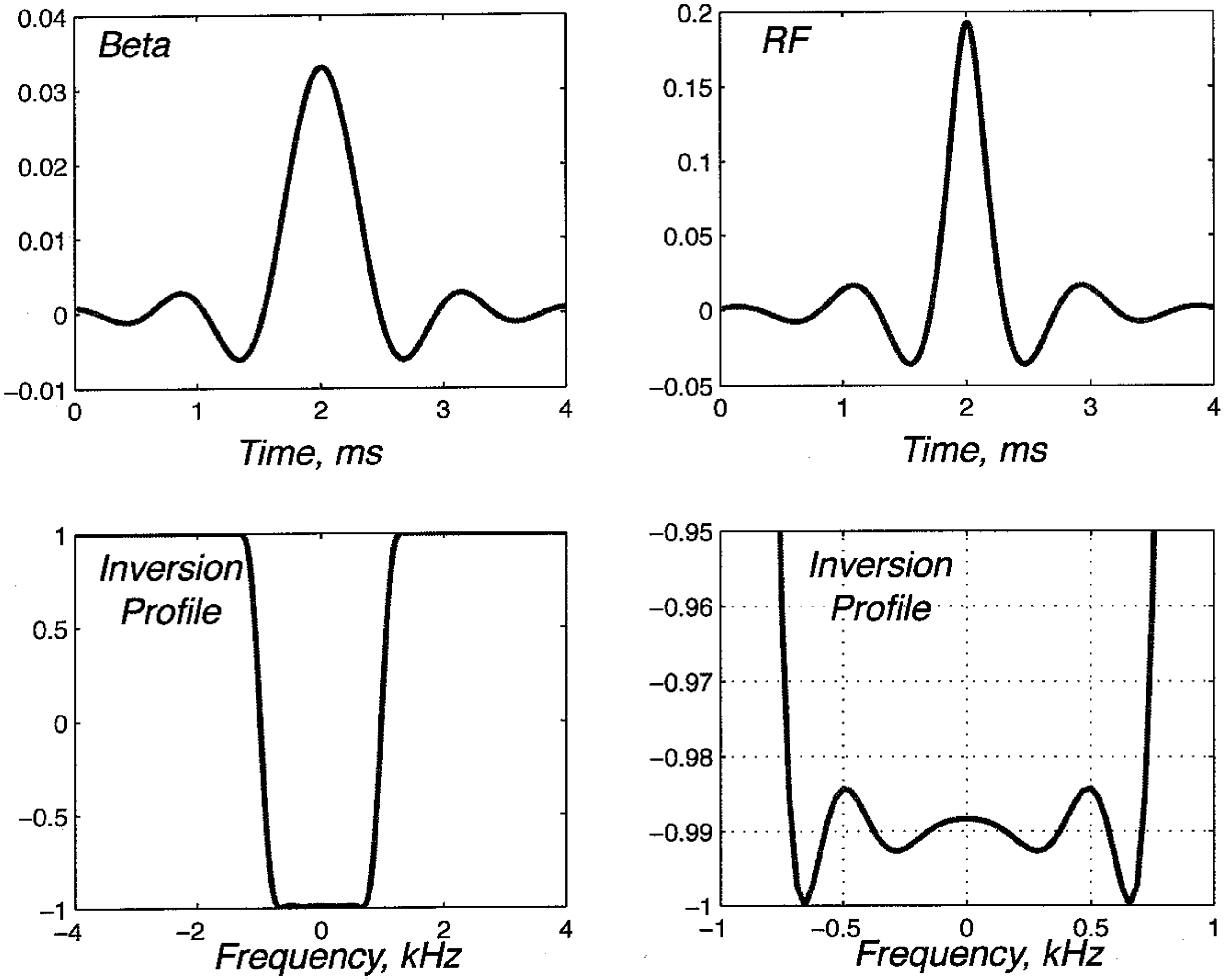
PM vs LS Filter Design

Time-Bandwidth = 8, 1% pass/stopband ripples
Same band edges

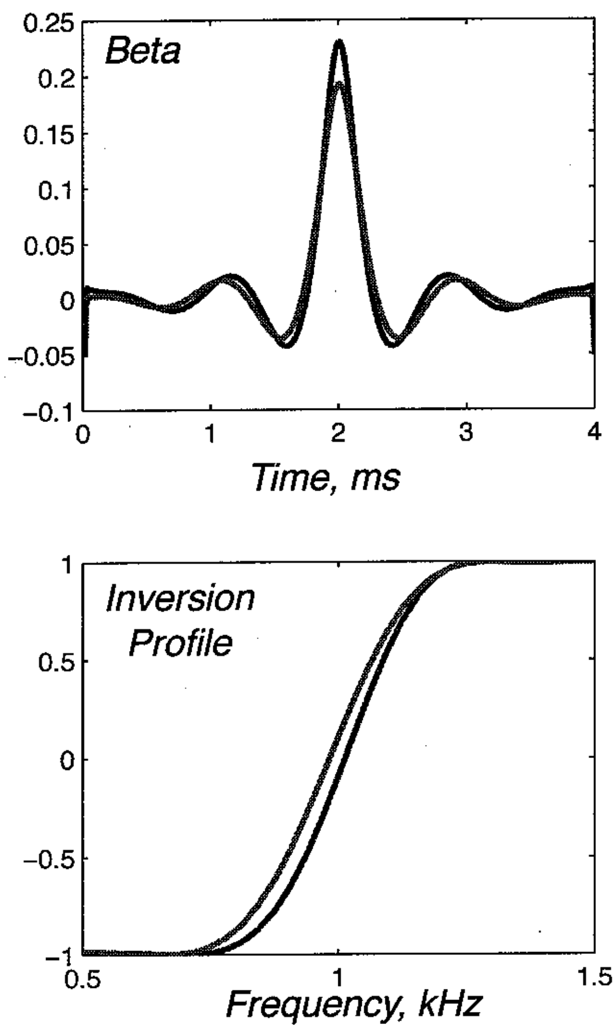


14a

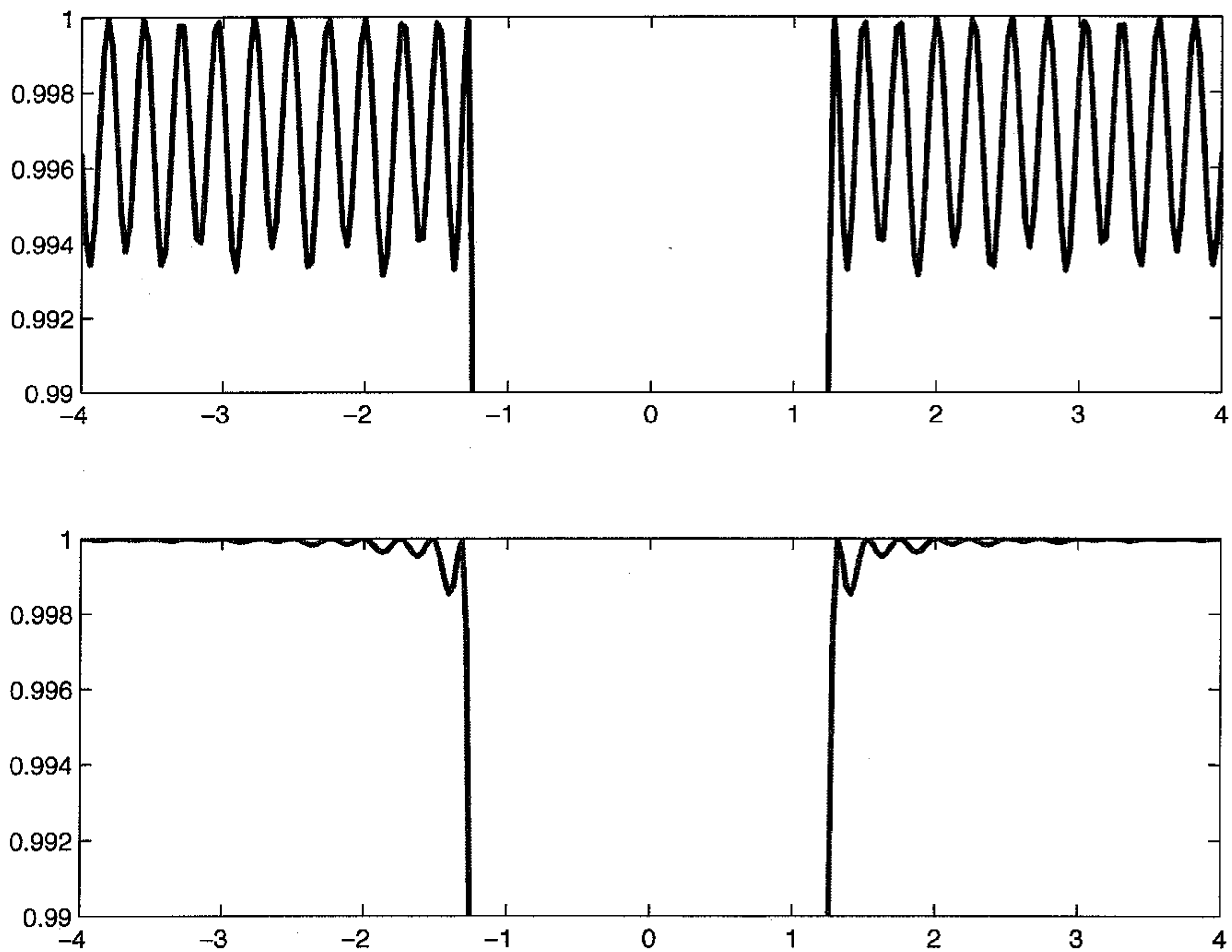
Weighted Least Squared Inversion



Comparison Between PM and LS Inversions

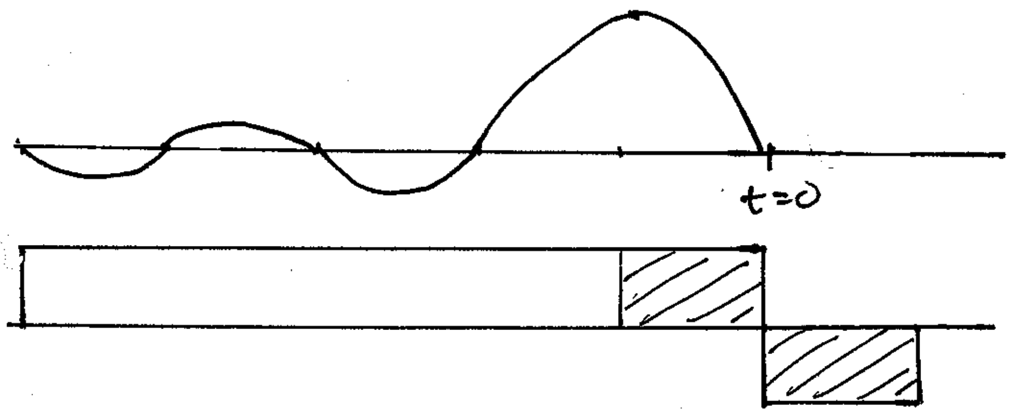


Comparison Between PM and LS Inversions



MINIMUM / MAXIMUM PHASE PULSES

EXCITATION AS LATE AS POSSIBLE



BENEFITS

SHARPER PROFILE

LESS REFOCUSING

SHORTER ECHO TIME

USES

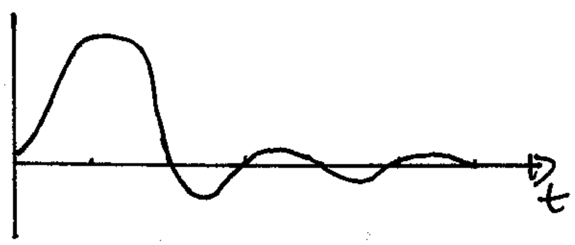
SLAB SELECT PULSE

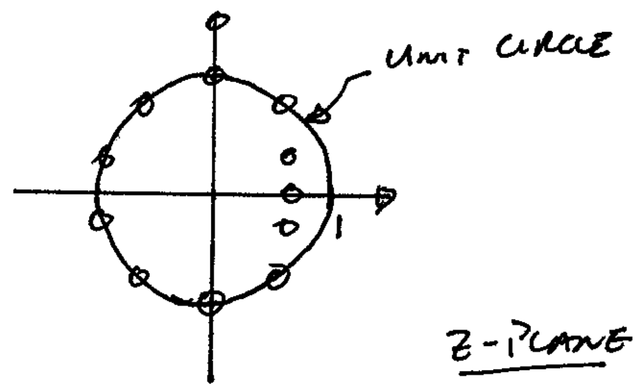
SATURATION PULSES

SHORT ECHO TIME EXCITATIONS

INVERSION PULSES

MINIMUM PHASE <u>CAUSAL</u> SIGNAL / FILTER





CAUSAL MINIMUM PHASE SIGNAL

SIGNAL CONCENTRATED AT BEGINNING

PASSBAND ZEROS INSIDE UNIT CIRCLE

MINIMUM PHASE RF PULSE

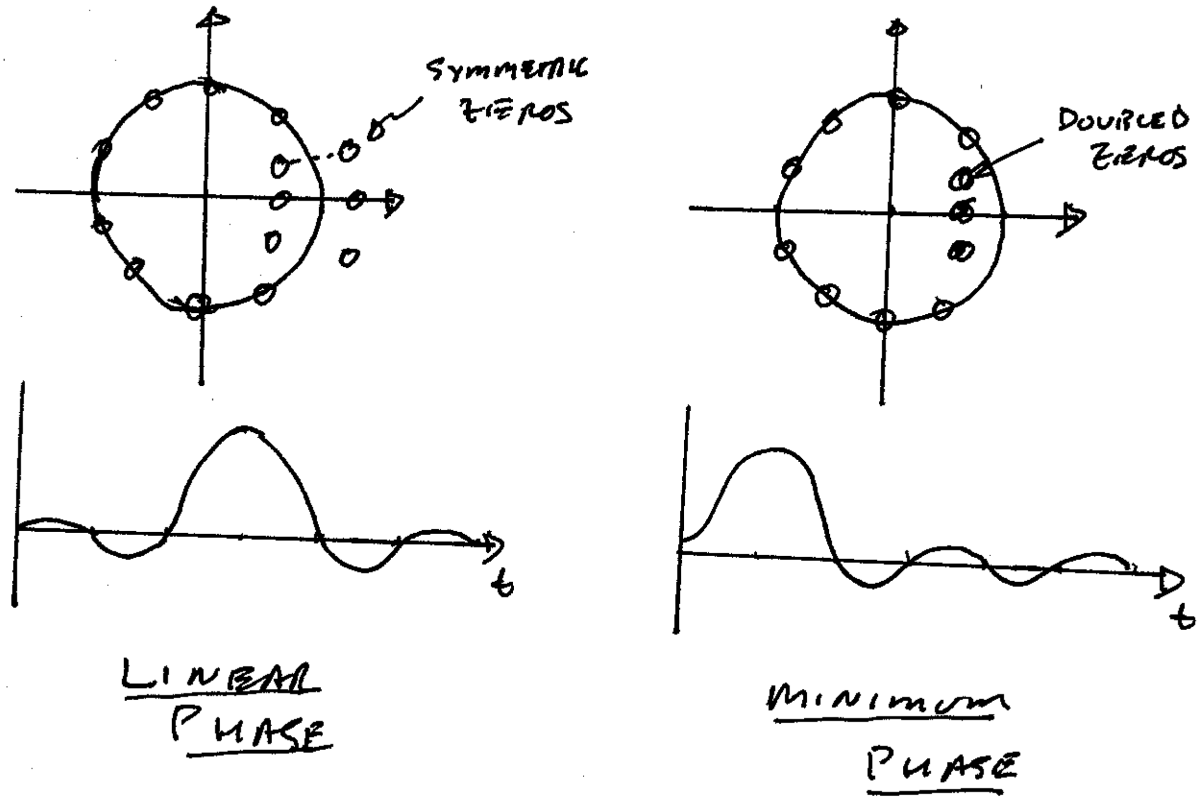
SIGNAL CONCENTRATED AT END (ORIGIN)

PASSBAND ZEROS OUTSIDE UNIT CIRCLE

CAUSAL DESIGN MUCH MORE FAMILIAR

DESIGN CAUSAL FILTERS, REVERSE FOR RF PULSE.

ANY SIGNAL HAS A MINIMUM PHASE SIGNAL WITH SAME MAGNITUDE



THESE BOTH HAVE SAME MAGNITUDE PROFILE!
NO GAIN IN SELECTIVITY.

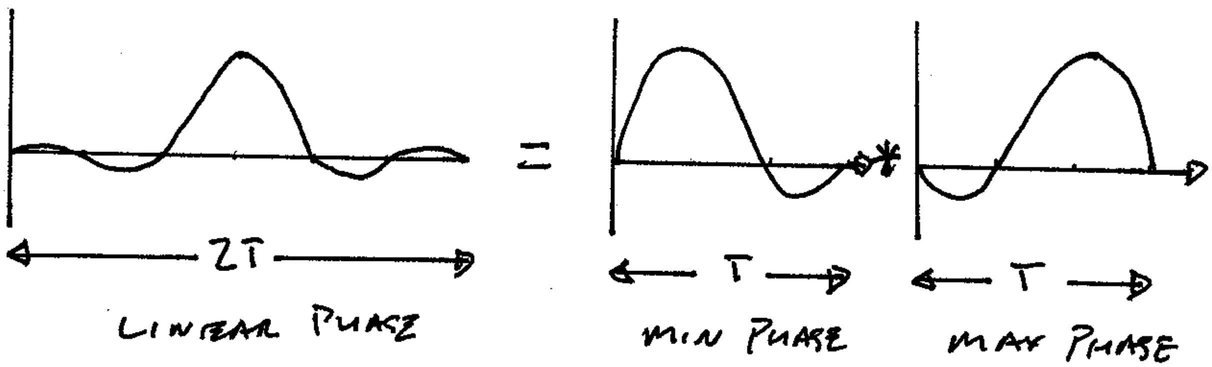
ONLY REALLY WANT SINGLE ZEROS INSIDE UNIT CIRCLE

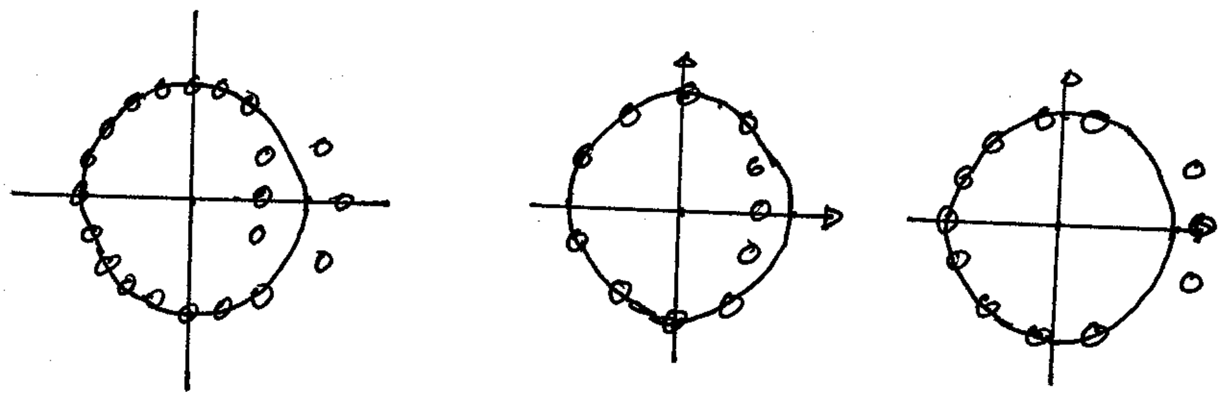
BASIC IDEA:

DESIGN A SPECIAL LINEAR PHASE PULSE

FACTOR INTO MINIMUM PHASE COMPONENT

LINEAR PHASE FILTER IS A CONVOLUTION OF A MINIMUM AND A MAXIMUM PHASE FILTER

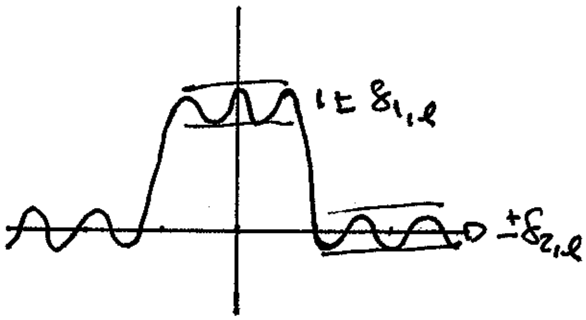


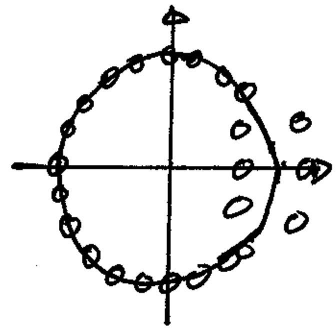


WE WANT TO DESIGN LINEAR PHASE FILTER TO BE EASY TO FACTOR

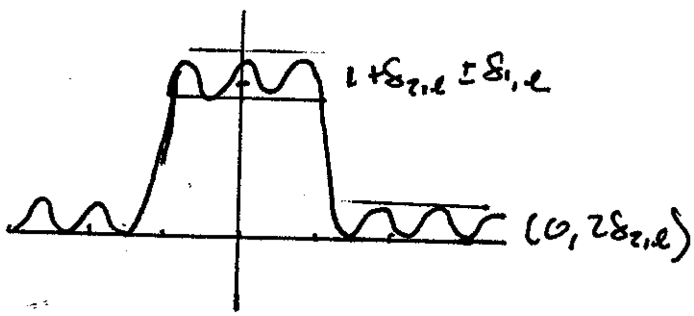
<u>EQUAL-RIPPLE (PARKS-McCLELLAN) FILTER</u>

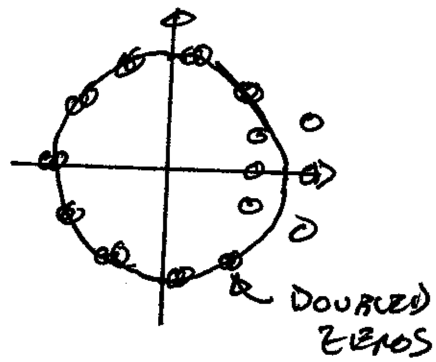
START WITH A PM FILTER





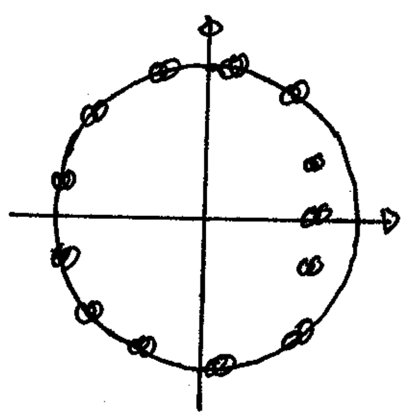
ADD A BIAS OF $\delta_{2,\ell}$





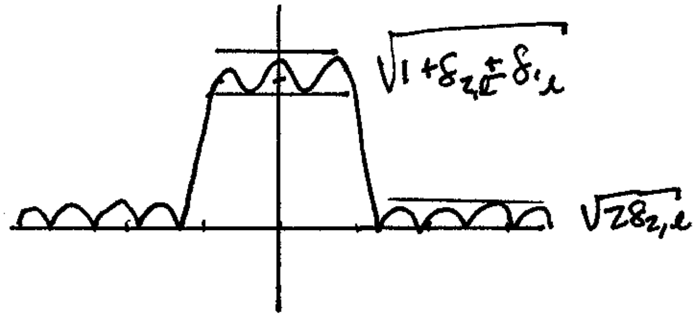
THIS MAGNITUDE PROFILE IS THE SAME AS

PERFECT SQUARE!

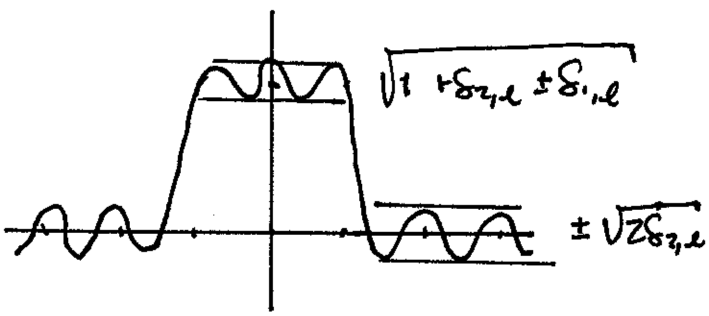


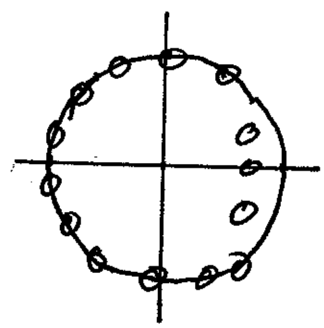
ALL ZEROS DOUBLED

TAKE SQUARE ROOT OF PROFILE



USE HILBERT TRANSFORM RELATIONSHIP TO FIND PHASE





EQUAL RIPPLE, MINIMUM PHASE PULSE

SHARPEST TRANSITION.

HOW DO WE DESIGN THE ORIGINAL LINEAR PHASE FILTER TO GIVE A SPECIFIED MINIMUM PHASE PROFILE?

PASSBAND RIPPLE

$$\sqrt{1 + \delta_{2,l} \pm \delta_{1,l}} \simeq 1 + \delta_{2,l} \pm \frac{1}{2}\delta_{1,l}$$

$$\delta_{1,m} \simeq \frac{1}{2}\delta_{1,l}$$

$$\underline{\delta_{1,l} = 2\,\delta_{1,m}}$$

STOPBAND RIPPLE

$$\delta_{2,m} = \sqrt{2\delta_{2,l}}$$

$$\underline{\delta_{2,l} = \delta_{2,m}^2 / 2}$$

DESIGN RELATION FOR LINEAR PHASE FILTER

$$(2T)(\Delta F) = D_\infty(\delta_{1,l}, \delta_{2,l})$$

$$\underbrace{(2T)}_{\text{LENGTH OF LINEAR PHASE FILTER}}(\Delta F) = D_\infty(2\delta_{1,m}, \delta_{2,m}^2/2)$$

WHERE

T - LENGTH OF MINIMUM PHASE FILTER

THEN

$$T\,\Delta F = \frac{1}{2}D_\infty(2\delta_{1,m}, \delta_{2,m}^2/2)$$

$$= D_{\infty,m}(\delta_1, \delta_2)$$

WHERE

$$D_{\infty,m}(\delta_1,\delta_2) = \frac{1}{2} D_{\infty}(2\delta_1, \delta_2^2/2)$$

RECALL

$$\Delta F = \frac{D_{\infty,m}(\delta_1,\delta_2)}{T}$$

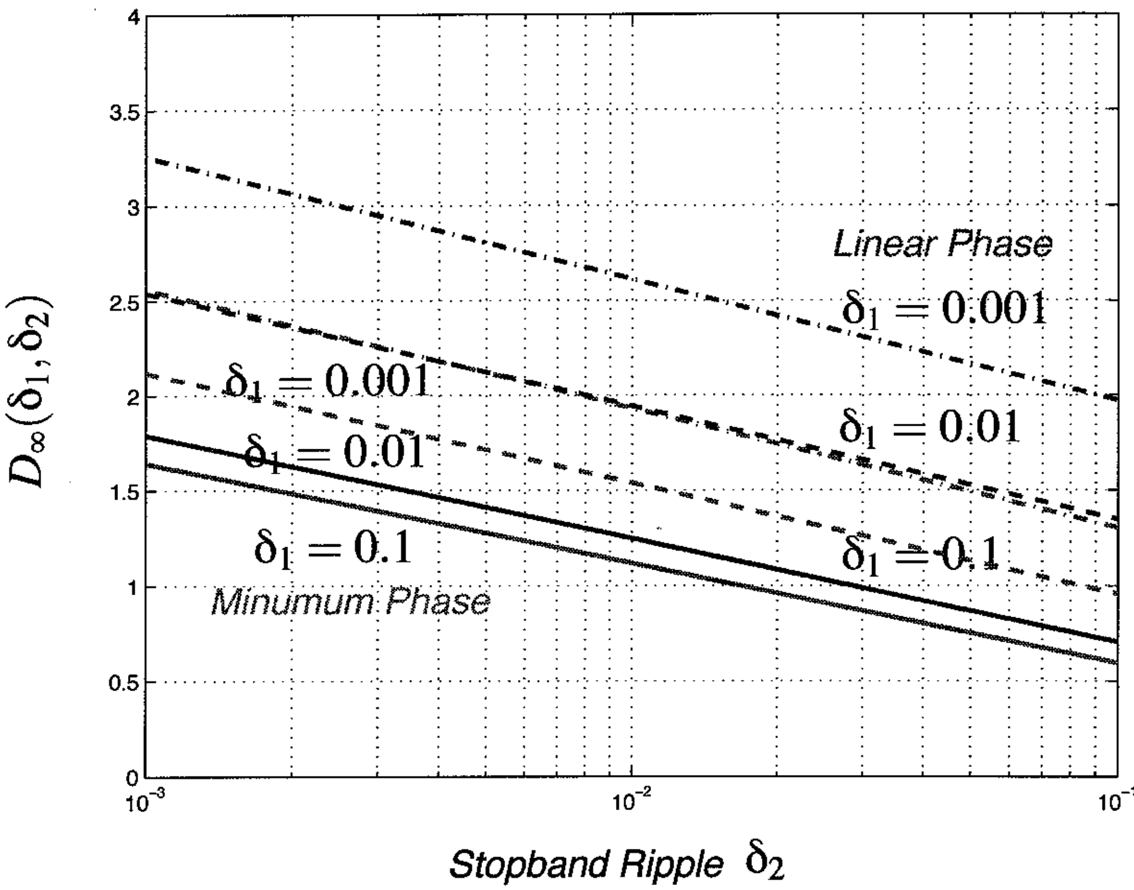
SO FOR A GIVEN T, A MINIMUM PHASE FILTER CAN HAVE HALF THE TRANSITION WIDTH OF LINEAR PHASE FILTER!

IN PRACTICE, THIS IS LESS.

TYPICAL NUMBERS ARE 70-90%

INCREASES WITH T (BW)

$$D_\infty(\delta_1, \delta_2) \quad \textit{vs} \quad D_{\infty,m}(\delta_1, \delta_2)$$



$$D_{\infty,m}(\delta_1, \delta_2) = \frac{1}{2} D_\infty\left(2\delta_1, \delta_2^2/2\right)$$

TYPICAL TRADEOFFS

FOR ANY δ_2, WE CAN IMPROVE δ_1 FROM 0.01 TO 0.001

FACTOR OF TEN IN PASSBAND RIPPLE

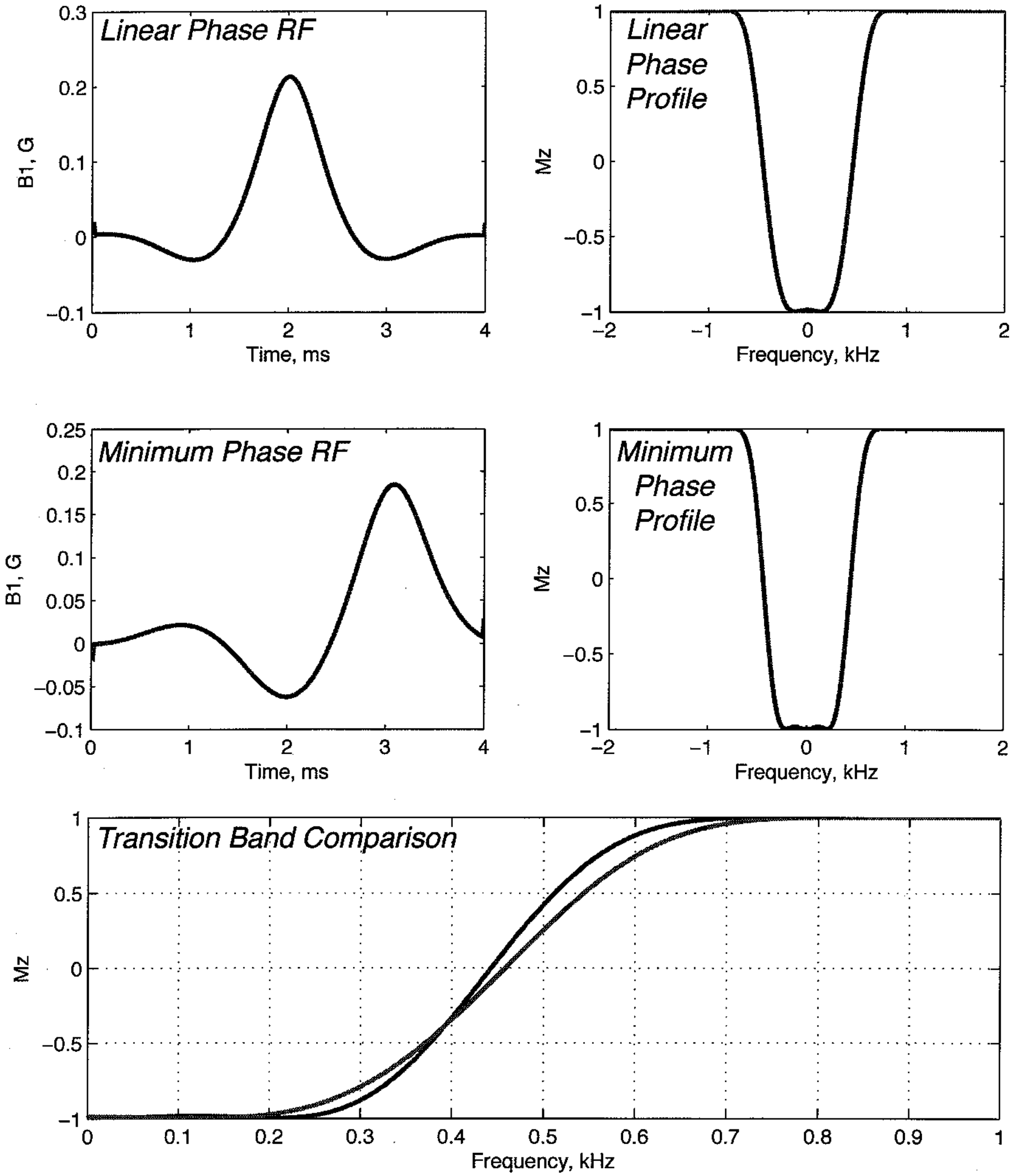
SIMILARLY, FIX δ_1 AND IMPROVE STOPBAND RIPPLE BY FACTOR OF TEN

FIX δ_1 AND δ_2 AND REDUCE TRANSITION WIDTH

IF $\delta_2 = 0.001$, $\delta_1 = 0.001$, D_∞ GOES FROM 2.6 TO 2. W REDUCED TO 75%.

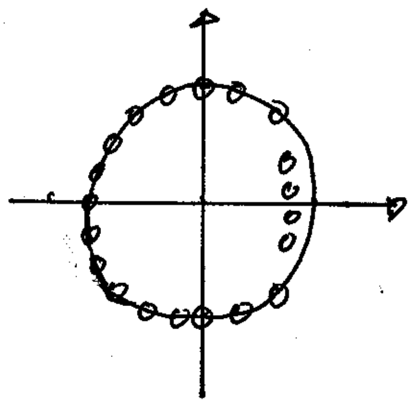
Linear vs Minimum Phase Inversion Pulses

$T(BW) = 4,\ \delta_1 = 0.01,\ \delta_2 = 0.0001$

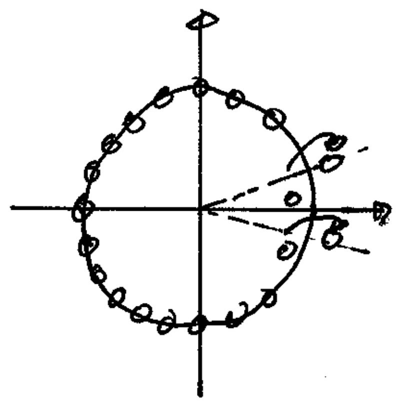


OTHER PHASE PROFILES

ONCE WE HAVE A MINIMUM PHASE DESIGN, THERE ARE MANY OTHER PHASE PROFILES THAT HAVE THE SAME MAGNITUDE PROFILE



MINIMUM PHASE



NON-LINEAR PHASE

EACH PASSBAND ZERO MAY BE FLIPPED OUTSIDE UNIT CIRCLE

THERE ARE ABOUT $T(BW)$ PASSBAND ZEROS

$$N_P \cong T(BW)$$

SO THERE ARE

$$2^{N_P}$$

POSSIBLE PHASE PROFILES

IF PROFILE PHASE IS NOT A CONCERN (SAT PULSES, INVERSION PULSES) WE CAN CHOOSE PHASE TO OPTIMIZE SOME OTHER PARAMETER

$\Rightarrow$ PEAK RF AMPLITUDE

DESIGN PROCEDURE

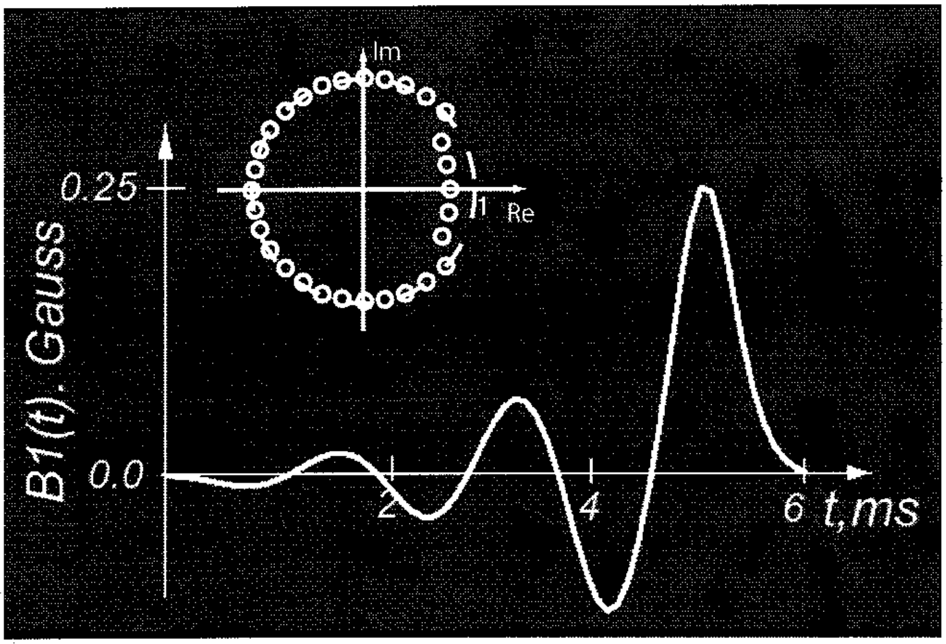
1) DESIGN MINIMUM PHASE $B_N(Z)$
2) FACTOR (ROOTS.M IN MATLAB)
3) CHECK EACH COMBINATION OF ROOT FLIPS
   a) CALCULATE $B_N(Z)$
   b) DESIGN RF PULSE
4) CHOOSE SOLUTION WITH MINIMUM PEAK $B_1(t)$.

CURRENTLY WORKS FOR 18 PASSBAND ZEROS, IN TENS OF SECONDS OF CPU TIME.

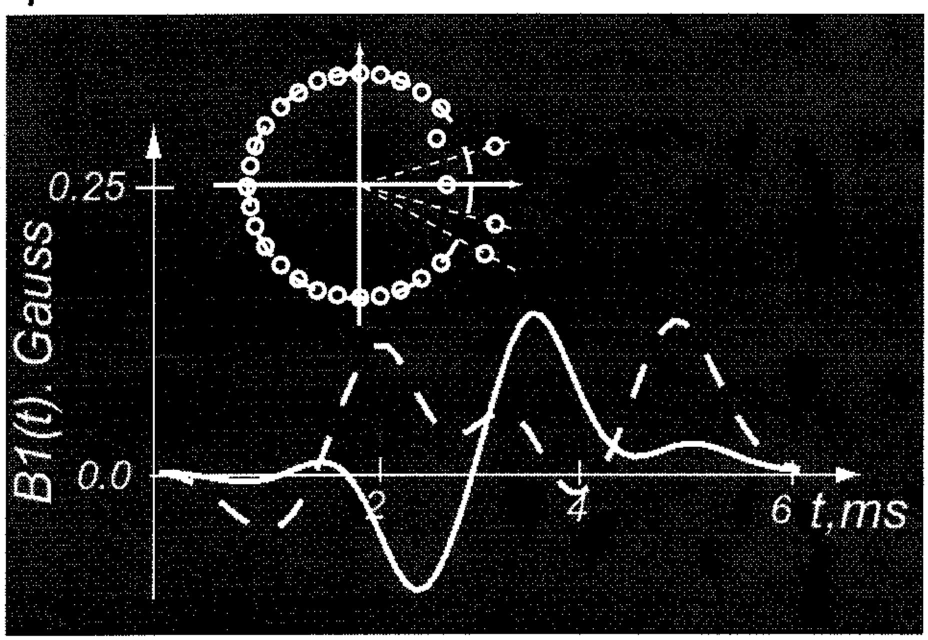
WON'T GET YOU TO 32 PASSBAND ZEROS SOON!

Non-Linear Phase Inversion Pulses

Minimum Phase Inversion



Optimized Phase Inversion



Peak Amplitude reduced by a factor of 2,
Peak power by a factor of 4